CLAVIUS'S COMMENTARY ON THE SPHERICKS OF *THEODOSIUS Tripolitæ*: OR, Spherical Elements,

Necessary in all Parts of MATHEMATICKS, wherein the Nature of the *Sphere* is considered.

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Clavius's *Preface.*

BEcause Geographers and Historians have described two Cities; the one in Phœnicia, and the other in Africa, both called by the Name of Tripolis, Writers are not certain whether Theodosius was a Phœnician, or an African. They differ also about the Time wherein he flourished: But it is very probable, he liv'd about the Time of Pompey the Great: Because Strabo says, he was Cotemporary with Asclepiades the Physician, in Bythinia, who, if we may credit Pliny, flourished in the Time of Pompey the Great. He wrote various small Ma-

thematical Tracts, as De Habitationibus, De Noctibus, & Diebus, *and likewise these three learned Books of Sphericks; in which he has demonstrated diverse Properties of the Sphere, the Knowledge of which is absolutely necessary in Astronomy. For without these Astronomy could not maintain its Dignity. Likewise Dialling very much depends on the Knowledge of these Sphericks; as also they are of great Use in rightly understanding of Geography, and Prospective,* &c.

And because there are extant two Versions of Theodosius's Sphericks; *the one being* John Pena's, *copy'd from the Original Greek; and the other* Maurolycus's, *taken from the Tradition of the* Arabians : *I think it proper to follow the former, in which are contained fifty Propositions, and lay down various* Scholia, *by which we demonstrate several necessary and plea-*

pleaſant Theorems, omited by Theodoſius, *but added by the* Arabians. *We did not think it proper in the Demonſtrations to follow the Words of the Greek Book, but the Senſe, that ſo the Demonſtrations might be more conſpicuous. We have likewiſe here and there added certain* Corollaries, Scholia, *and* Lemmata, *to be uſed when there is Occaſion for them. Moreover, we have moſtly neglected the Figures in the Greek Copy, becauſe thoſe in* Maurolycus's *are more proper and eaſier to be underſtood. Laſtly, that the Courſe of the Demonſtration might not be interrupted, we have cited the Propoſitions of* Euclid, *and of theſe Books in the Margin.*

The Citations are thus to be underſtood.

1. 1. The firſt Prop. of *lib.* 1. *Eucl.*
Cor. 16. 3. The Corollary of Prop. 16. *lib.* 3. *Eucl.*
4. of this. The 4*th* Prop. of this Book.
12. 2. of this. Prop. 12. of *lib.* 2. of this Work.

Adver-

Advertiſement.

THE

THE Spherical Elements OF *THEODOSIUS.*

BOOK I.

DEFINITIONS.

I.

A Sphere is a ſolid Figure contain'd under one Superficies, to which from one Point within it, all Right Lines that be drawn, are equal between themſelves.

II.

The aforeſaid Point, is called the Center of the Sphere.

III.

The Axis of a Sphere, is a Right Line drawn thro' the Center, and terminated on both Sides by the Superficies of the Sphere, about which the Sphere revolves.

IV.

The Poles of a Sphere, are the Extremes of its Axis.

V.

The Pole of a Circle in a Sphere, is a Point in its Superficies, from which all Right Lines drawn to the Circumference of the Circle are equal to one another.

SCHOLIUM.

There is yet added, in the Greek Verſion, another Definition, explaining what is meant by the Similar Inclination of Plans. But becauſe the Inclination of Plans is explained by Euclid,*in* Lib. 11. Def. 6.*and their Similar Inclination in* Def. 7. *of the ſame Book, I have here omitted it, and inſtead thereof put the following Definition, not much unlike* Def. 4. Lib. 3: Euclid.

VI.

Circles in a Sphere, are ſaid to be equally diſtant from the Center, when Perpendiculars, let fall from the Center of the Sphere, to the Plans of the Circles, are equal between themſelves: And that Circle is ſaid to be furtheſt diſtant, when the Perpendicular drawn to its Plan is greateſt.

THEO. I. PROP. I.

If the Superficies of a Sphere be cut by any Plan, the Line made in its Superficies, is the Circumference of a Circle.

Fig. 1. LET the Spherical Superficies ABC, whoſe Center is D, be cut by any Plan, making in the Superficies of the

Sphere

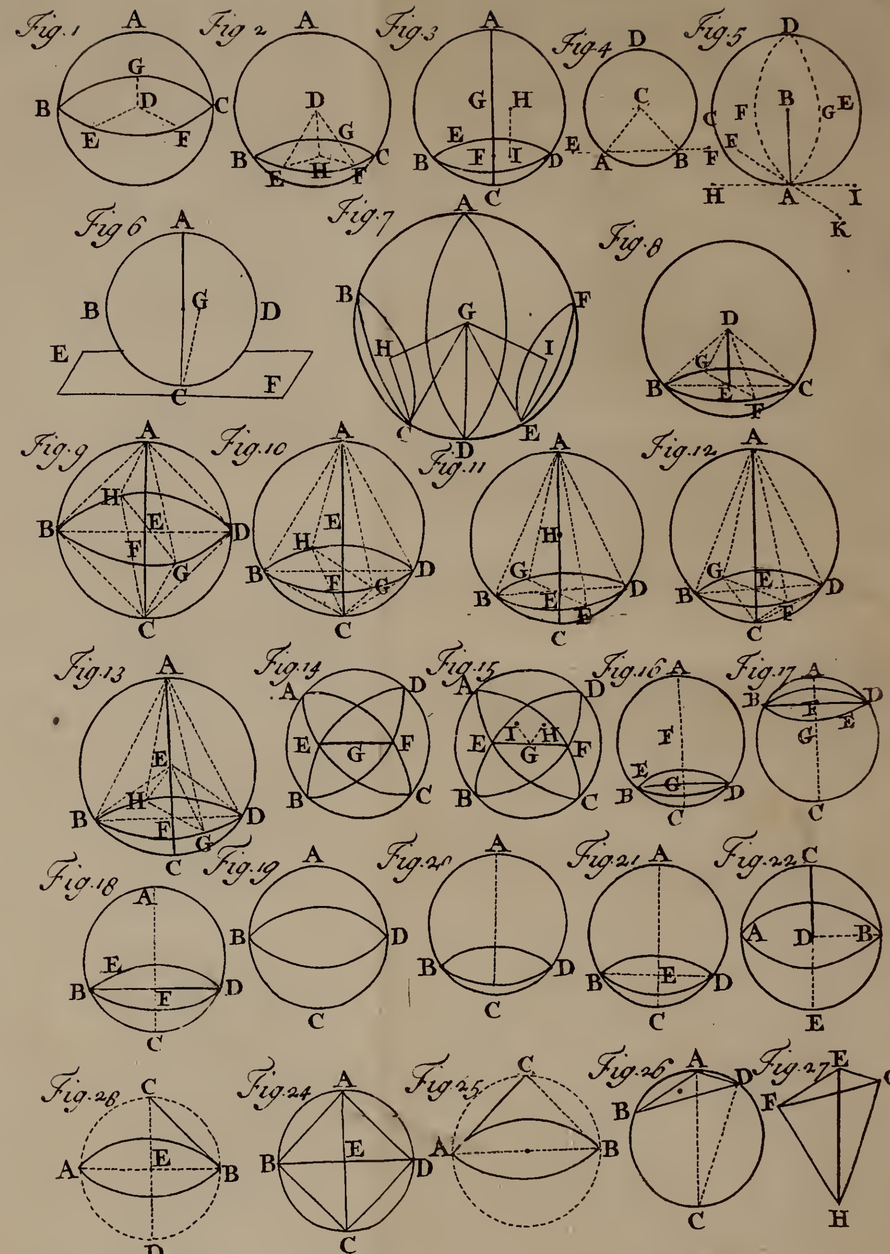


Fig. 1
Fig. 2
Fig. 3
Fig. 4
Fig. 5
Fig. 6
Fig. 7
Fig. 8
Fig. 9
Fig. 10
Fig. 11
Fig. 12
Fig. 13
Fig. 14
Fig. 15
Fig. 16
Fig. 17
Fig. 18
Fig. 19
Fig. 20
Fig. 21
Fig. 22
Fig. 24
Fig. 25
Fig. 26
Fig. 27

Sphere the Line BEFCG. I ſay BEFCG, is the Circumference of a Circle. For, firſt let the Plan paſs thro' the Center D of the Sphere, ſo that D may be in the ſaid Plan, in which, from D to the Section BEFCG, draw any number of right Lines, as DE, DF, DG. Therefore becauſe all theſe Lines, be they never ſo many, drawn from the Center of the Sphere to its Superficies, are equal to each other, the Line BEFCG (by *Def.* 15. *lib.* 1. *Euclid*,) will be the Circumference of a Circle, whoſe Center is D, the ſame as the Center of the Sphere.

2*dly*. Let the cutting Plan not paſs thro' the Center of the Sphere, (*a*) and draw from D, the Centre of the Sphere, to the Plan, the Perpendicular DH; draw likewiſe from H, right Lines, as HE, HF, any how, to the Line BEFCG, and join the right Lines DE, DF. Therefore becauſe the Angles DHE, DHF, are right ones (from *Def.* 3. *lib.* 11. *Euclid*,) (*b*) the Square of ED, is equal to the Squares of DH, HE, and the Square of DF, to the Squares of DH, HF: But the Squares of DE, DF, are equal to each other, becauſe the right Lines DF, DE, drawn from the Center of the Sphere to its Superficies, are equal: Therefore the Squares of DH, HE together, are equal to the Squares of DH, HF together. From whence taking away the common Square of the right Line DH, the remaining Squares of the right Lines HE, HF, are equal to one another, and accordingly the right Lines HE, HF, will be equal to each other. In the ſame manner may it be demonſtrated, that all right Lines drawn from H, to the Line BEFCG, are equal between themſelves, and to the ſaid two Lines HE, HF. Therefore the Line BEFCG, will be the Circumference of a Circle, (from *Def.* 15. *lib.* 1. *Euclid*,) whoſe Center is the Point H, in which the Perpendicular falls. *Q. E. D.*

Fig. 2.
(*a*) 11. 11.
(*b*) 47. 1.

COROLLARY.

Therefore if the cutting Plan paſſes thro' the Center of a Sphere, there will be a Circle made, having the ſame Center with the Center of the Sphere. But if it does not paſs thro' the Center of the Sphere, there will be a Circle made, not having the ſame Center as that of the Sphere. But having that Point for its Center, in which the Perpendicular, drawn from the Sphere's Center to the cutting Plan, falls.

That is,

The Center of a Sphere, is the same with the Center of a Circle passing thro' the said Center, and a Perpeudicular drawn from the Center of a Sphere, to the Plan of a Circle not passing thro' the Center of the Sphere, falls in the Center of the Circle: Because the Point H in which the Perpendicular DH, falls, has been proved to be the Center of the Circle.

PROB. I. PROP. II.

To find the Center of a given Sphere.

Fig. 3. IT is required to find the Center of the Sphere ABCD. Cut its Superficies by any Plan, whose Section suppose BDE, *(a)* which will be the Circumference of a Circle; *(b)* let the Center of this Circle be F. If therefore the Circle BDE, passes thro' the Center of the Sphere, the point F, *(c)* will be also the Center of the Sphere. But if the Circle does not pass thro' the Center of the Sphere, *(d)* raise from F, to the Plan of the Circle BDE, the Perpendicular FG, which produced both ways to the Superficies of the Sphere in the Points A, B, and being bisected in the Point G. I say G, is the Center of the Sphere: For if it is not, let H be the Center, cutting all the Diameters in half, which will not be in the Line A C, because that is only bisected in the Point G, but without it. *(e)* Draw from H the Center of the Sphere, to the Plan of the Circle BDE, the Perpendicular HI, *(f)* which will be parallel to FG; and accordingly will not fall in the Point F: for then two Parallels GF, HI, would meet in the Point F, which is impossible. But because the Perpendicular drawn from the Center of the Sphere to the Plan of the Circle BDE, *(g)* falls in its Center, I will be the Center of the Circle BDE. But likewise F, from Construction, is the Center of the same Circle; which is absurd: for the

(a) 1. *of this.*
(b) 1. 3.
(c) Cor. 1. *of this.*
(d) 12. 11.
(e) 11. 11.
(f) 6. 11.
(g) Cor. 1. *of this.*

the ſame Circle hath only one Center, therefore no other Point beſides G, will be the Center of the Sphere. Q. E. F.

COROLLARY.

From hence it is manifeſt, that if there is a Circle in a Sphere not paſſing thro' the Center of the Sphere, from whoſe Center is raiſed a Perpendicular to its Plan, the Center of the Sphere, will be in that Perpendicular, for it has been demonſtrated that the Point G biſecting the Perpendicular AC, is the Center of the Sphere.

THEO. II. PROP. III.

A Sphere doth not touch a Plan, by which it is not cut, in more Points than One.

FOR if it can be, let a Sphere touch a Plan, by which it is not cut, in more Points than One, as in A, B. now *(a)* C the Center of the Sphere, being found, draw the right Lines CA, CB: and thro' CA, CB draw a Plan making in the Superficies of the Sphere *(b)* the Circumference of the Circle ABD, *(c)* and touching the right Line EABF in the Plan. Therefore becauſe the touching Plan, in which the right Line EABF is, does not cut the Sphere, neither the Circle ADB in its Superficies, it's manifeſt the right Line EABF, will not cut the Circle ABD. Therefore the right Line ABD, will fall quite without the Circle. But becauſe the two aſſumed Points A, B, are in the Circumference of the Circle ABD, *(d)* the ſame right Line AB, drawn from the Point A to the Point B, will fall quite within the Circle ABD; which is abſurd. Therefore a Sphere cannot touch a Plan, by which it is not cut, in more Points than One. Q. E. D.

Fig. 4.

(a) 2 *of this.*

(b) 1 *of this.*

(c) 3. 11.

(d) 2. 3.

COROL-

COROLLARY.

Hence, if two Points are assigned in the Superficies of a Sphere, a right Line joyning them will fall within the Sphere. *(e)* Because it falls within a Circle whose Circumference is in the Sphere's Superficies.

(e) 2. 3.

THEO. III. PROP. IV.

If a Sphere touches a Plan, which does not cut it, a right Line drawn from the Center of the Sphere to the Point of Contact, will be perpendicular to the Plan.

Fig. 5.
(a) 2. of this.
(b) 1. of this.
(c) 3. 11.
(d) Cor. 1. of this.
(e) 18.

LET a Sphere touch a Plan, not cutting of it, in the Point A : (*a*) and the Center B of the Sphere being found, draw from it to the Point of Contact A, the Line BA. I say the Line BA is perpendicular to the said Plan. For draw two Plans any how thro' the Line AB mutually cutting each other, which (*b*) make the Circumferences ACDE, AFDG, of Circles, in the Superficies of the Sphere, and (*c*) touching the right Lines HAI, KAL, in the Plan. Therefore because both the Circles ACDE, AFDG, pass thro' the Center B of the Sphere, (*d*) B will be the Center of them both. Again, because the Plan touches the Sphere, and does not cut it, neither will the right Lines HAL, KAL, which are in it, cut the same, and accordingly neither the Circles ACDE, AFDG, existing in the Sphere's Superficies. Therefore the right Line HAI, touches the Circle ACDE, in the Point A, and the right Line KAL, the Circle AFDG, in the same Point A. (*e*) Therefore the right Line BA, is both perpendicular to HAI, and KAL. Whence the right Line BA, will be perpendicular to the Plan of Contact, drawn thro' the right Lines HAI, KAL. Q. E. D.

THEO. IV. PROP. V.

If a Sphere touches a Plan, which does not cut it, and from the Point of Contact is raised a right Line perpendicular to the Plan, the Center of the Sphere will be in the said Perpendicular.

LET the Sphere ABCD, touch the Plan EF, which does not cut it, in the Point C, and let there be (*a*) raised to the Plan EF, the Perpendicular CA. I say the Center of the Sphere is in the right Line AC. For if it is not, let the Center of the Sphere be without the Line AC, and draw a right Line from G to C, (*b*) which will be perpendicular to the Plan AC. Therefore from the same Point C to the same Plan EF are two Perpendiculars drawn; which is absurd: for two right Lines cannot (*c*) be raised at right Angles in a given Plan, from a Point given in it. Q. E. D.

Fig. 6.
(*a*) 12. 11.
(*b*) 4. *of this.*
(*c*) 13. 11.

THEO. V. PROP. VI.

The greatest Circles drawn in a Sphere, are those passing thro' its Center: And those which are equally distant from the Center, are equal: But those which are further distant from the Center are lesser. And contrarywise, great Circles in a Sphere pass thro' its Center: Those that are equal are equally distant from the Center: But those are lesser, that are further from the Center of the Sphere.

LET the Circle AD, pass thro' the Center G, of the Sphere ABCDEF, and the others BC, EF not thro' the

Fig. 7.

the Center. I say AD is a Circle the greatest of all, &c.
(a) 11. 11. For (*a*) draw the Perpendicnlars GH, GI, from the Center G, to the Plans of the Circles BC, FE,
(b) Cor. 1. *of this.* which (*b*) will fall in their Centers; so that H, I, will be the Centers of the Circles BC, EF: (*c*)
(c) Cor. 1. *of this.* but G the Center of the Sphere, is also the Center of the Circle, AD, passing thro' the Sphere's Center. If therefore from G, H, I, to the Superficies of the Sphere are drawn the right Lines, GD, HC, IE, these will be the Semidiameters of the Circles AD, BC, FE. Also join the right Lines GC, GE. Therefore because in the Triangle GHC, the Angle H, is a right one (*per Def.* 3. *lib.* 11. *Euclid*) (*d*) the Square of GC
(d) 47. 1. will be equal to the Squares of GH, HC. Whence taking away the common Square of the right Line GH, the Square of GC, will be greater than the Square of HC; and therefore likewise the right Line GC, that is, GD, (for GC, GD are drawn from the Center of the Sphere to its Superficies) is greater than the right Line HC. Whence the Circle AD having a greater Semidiameter than the Circle, BC will be greater than the Circle BC. By the same Way of Reasoning we may demonstrate, that the Circle AD is greater than any other not drawn thro' the Center. Therefore the Circle AD, is the greatest.

Now let the Circles BC, EF, be equally distant from the Center G, that is, let the Perpendiculars GH, GI, be equal, from *Def.* 6. of this Book. I say the Circles BC, EF, are equal. For when the right Lines GC, GE, falling from the Center of the Sphere to its Superficies, are
(e) 47. 1. equal, and accordingly their Squares equal; (*e*) and also the Square of GC equal to the Squares of GH, HC, and the Square of GE equal to the Squares of GI, IE; the Squares of GH, HC together, will be equal to the Squares of GI, IE, together. Therefore taking away the equal Squares of the right Lines GH, GI, (for these Lines are supposed equal) the remaining Squares of the right Lines HC, IE, will be equal, and accordingly also the right Lines HC, IE, will be equal: But when they are the Semidiameters of the Circles BC, FE, these Circles will likewise be equal.

If one of the Circles, *viz.* BC, is placed further distant from the Center than the other FE, that is, if the perpendicular GH be supposed greater than GI, we

may

may demonſtrate almoſt in the ſame manner, that the Circle BC is leſſer than the Circle FE, for ſince the Squares of GH, H,C have been demonſtrated to be equal to the Squares of GI, IE; If the unequal Squares of the unequal right Lines GH, GI are taken away, (the Square of GH being greater than the Square of GI,) the remaining Square of the right Line HC, will be leſſer than the remaining Square of the right Line IE; and accordingly alſo the right Line HC, will be leſſer than the right Line IE. And therefore the Circle BC, will be leſſer than the Circle FE.

Now let AD be the greateſt Circle of all. I ſay it paſſes thro' G, the Center of the Sphere. For if it do not paſs thro' the Center, ſome other Circle paſſing thro' the Center, will be greater than the Circle AD, not paſſing thro' the Center, as has been demonſtrated in this Propoſition. Therefore AD, is not the greateſt Circle: Which is abſurd. For it is poſited the greateſt. Therefore it paſſes thro' G, the Center of the Sphere.

Again, let the Circles BC, FE, be equal. I ſay they are equally diſtant from G, the Center of the Sphere. For the Fignre being conſtructed as before, the Semidiameters HC, IE, will be equal. And becauſe the Squares of GH, HC, are equal to the Squares of GI, IE, (*f*) as has been demonſtrated; the equal Squares of the equal Lines HC, IE, being taken away, the remaining Squares, of the right Lines GH, GI, will be equal; and accordingly alſo the right Lines GH, GI, will be equal, which when they are perpendicular, from Conſtruction, to the Plans of the Circles BC, FE, the Circles, BC, FE, will be equally diſtant from the Center G, from *Def.* 6. of this Book.

(*f*) 47. 1.

Laſtly, If one of the Circles BC, FE, *viz.* BC, be leſſer than the other Circle FE, it may in the ſame manner, be demonſtrated, that the Perpendicular GH, is greater than the Perpendicular GI. For becauſe the Squares of GH, HC, have been proved to be equal to the Squares of GI, IE; and the Square of HC, being leſſer than the Square of IE; (becauſe from the Hypotheſis, the Semidiamiter HC, of the leſſer Circle, is leſſer than the Semidiameter IE, of the greater Circle) the remaining Square of the right Line GH, will be greater than the remaining Square of the right Line GI; and

C there-

therefore alſo the right Line GH, will be greater than GI. Wherefore ſince GH, GI, are perpendicular, from Conſtruction, to the Plans of the Circles, the leſſer Circle BC, will be further diſtant (*Def.* 6. of this Book) from the Center G, than the greater Circle FE. Q. E. D.

THEO. VI. PROP. VII.

If there is a Circle in a Sphere, and from the Center of the Sphere to the Center of the Circle a right Line is drawn; the ſaid Line, will be Perpendicular to the Plan of the Circle.

Fig. 8. IN the Sphere ABC, whoſe Center is D, let there be a Circle, as, BFCG, whoſe Center is E, and let the right Line DE, connect their Centers D, E: I ſay the right Line DE, is perpendicular to the Plan of the Circle BFCG. For having any how drawn the two Diameters BC, FG, in the Circle, draw from their Extremes, to D the Center of the Sphere, the right Lines BD, CD, FD, GD, which will be all equal to one another, as being drawn from the Center of the Sphere to its Superficies: alſo BE, CE, FE, GE, the Semidiameter of the Circle BFCG, are equal. Therefore the two Triangles DEB, DEC, have two Sides DE, EB, equal to two ſides DE, EC, as alſo the Baſe DB equal to the Baſe DC; whence the Angles DEB, DEC, *(a)* are
(*a*) 8. 1. equal and therefore right ones. Wherefore the right Line DE, is Perpendicular to the right Line BC.

In the ſame manner may it be proved, that the right Line DE, is Perpendicular to FG. *(b)* Therefore alſo it
(*b*) 4. 11. will be Perpendicular, to the Plan of the CircleBFCG, drawn thro' the right Lines BC, FG. Q. E. D.

THEO.

THEO. VII. PROP. VIII.

If there is a Circle in a Sphere, and from the Center of the Sphere to the Circle be drawn a Perpendicular: The said Perpendicular produced both ways, will fall in the Poles of that Circle.

IN the Sphere ABCD, whose Center is E, let there be the Circle BGDH, in the Plan of which from the Sphere's Center let there be *(a)* drawn a Perpendicular, as EF, which both ways produceed falls in the Superficies of the Sphere, at the Points A, C. I say, A, C, are the Poles of the Circle BGDH. For the Perpendicular EF, falls in the Center of the Circle BGDH, and therefore F, will be the Center of the Circle. Now if the Circle BGDH, is drawn thro' the Center of the Sphere, *(b)* the Center E of the Sphere, will be the same, with the Center F of the Circle, *(c)* from which to the Plan of the Circle let the Perpendicular AC be raised. Therefore the Diameters BD, GH, being any how drawn, draw from their Extremes, right Lines to the Points A, C. And because AF is Perpendicular to the Plan of the Circle BGDH, all the Angles made at F, will be right ones (from *Def.* 3. *Lib.* 11. *Euclid.*) Wherefore the two Triangles AFB, AFH, have two sides AF, FB, equal to two sides AF, FH, which comprehend equal Angles, *viz.* right ones. *(d)* Therefore the Bases AB, AH are equal. One may in the same manner, prove, that the right Lines AD, AG, or any others drawn from A to the Circumference of the Circle BGDH, are equal between themselves, and to the right Lines AB, AH. Therefore the Point A, is the Pole of the Circle BGDH, from *Def* 5. of this Book. By the same way of reasoning it may be demonstrated that C is also the Pole of the same Circle. Q. E. D.

Fig. 9. 10. *(a)* 11. 11. *(b)* Cor. 1. *of this.* *(c)* 12. 11. *(d)* 4. 1.

SCHOLIUM.

In the Version of Maurolycus *are annexed the two following Theorems, added by the* Arabians.

I.

If there is a Circle in a Sphere, from whose Center is raised a Perpendicular to the Plan of the Circle: This Perpendicular produced both ways, will fall in both the Poles of the Circle.

In the last Figure from F, the Center of the Circle
(*a*) 12. 11. *BGDH, (a) raise the right Line FA; perpendicular to the Plan of the Circle, cutting the Superficies of the Sphere, in the Points A, C. I say A, C, are the Poles of the Circle BGDH. For from* Def. 3. lib. 11. Euclid, *all the Angles which the right Line AF makes, at*
(*b*) 4. 1. *F, are right ones. (b) Wherefore, as before, the Lines AB, AD, AG, AH,* &c. *are equal to each other* &c.
Or otherwise thus. (c) Because the Perpendicular
(*c*) Cor. 2 *of this.* *FA passes thro' the Center E, of the Sphere, the right Line EF, drawn from E, the Center of the Sphere, will be Perpendicular to the Plan of the Circle BGDH.*
(*d*) 8. *of this.* *(d) Wherefore, as has been demonstrated, it falls in the Poles of the same Circle.*

II.

If there be a Circle in a Sphere, and from one of its Poles is drawn a right Line thro' it's Center; this Line, will be Perpendicular to the Plan of the Circle, and produced, will fall in the other Pole.

Still, in the same Figure, from A, the Pole of the Circle BGDH, draw the right Line AF, thro' its Center F, cutting the Superficies of the Sphere in the Point C. I say the right Line AF, is perpendicular to the Plan of the Circle BGDH, and C is the other Pole of the same Circle. For because the two Triangles

AFB, AFD, have two Sides, AF, FB, equal to two Sides AF, FD, and the Base AB equal to the Base AD, from the Def. of a Pole, the two Angles AFB, AFD, (a) will be equal, and therefore right ones. Whence the right Line AF, is perpendicular to BD. In the same manner, we demonstrate, that the same AF, is perpendicular to the right Line GH, (b) and consequently to the Plan of the Circle BGDH, drawn thro' the right Lines BD, GH. Which was the first thing to be demonstrated. Now because AF, is at right Angles to the Plan of the Circle BGDH, the right Line FA, drawn from the Center F, will be perpendicular to the Plan of the Circle. Wherefore, as has been just now demonstrated in this Scholium, if it be both ways produced, it will fall in each Pole of the Circle, and accordingly C, will be the other Pole of the Circle BGDH. Which was the second thing proposed. (a) 8. 1. (b) 4. 11.

THEO. VIII. PROP. IX.

If there be a Circle in a Sphere, and from one of its Poles, is drawn a Line Perpendicular to it: This Line will fall in the Center of the Circle, and from thence produced, will fall in the other Pole of the Circle.

IN the Sphere ABCD let there be the Circle BFDG, (a) from whose Pole A to its Plan, is drawn the Perpendicular AE, cutting the Superficies of the Sphere in C. I say E is the Center of the Circle BFDG, and C the other Pole. For having drawn thro' E two right Lines any how, as BD, FG, connect their Extremes, with the Pole A, by the right Lines AB, AD, AF, AG, which will be all equal, from the *Def.* of a Pole. Also all the Angles, that the right Line AE makes at E, will be right ones, from *Def.* 3. *lib.* 11. *Euclid.* (b) Therefore the Square of AB, will be equal to the Squares of AE, EB, and the Square of AG equal to the Squares of AE, EG; whence since the Squares of the equal Lines AB, AG, are equal, the (a) 11. 11. Fig. 11. (b) 47. 1.

Squares

Squares of AE, EB together, will be equal to the Squares of AE, GE, together. Therefore taking away the common Square of the right Line AE, the remaining Squares of the right Lines EB, EG, will be equal, and so the Lines themselves. In the same manner it may be demonstrated, that the right Lines EG, ED, are equal. (*c*) Wherefore E is the Center of the Circle BFDG; which was proposed. Therefore because from E, the Center of the Circle BEDG, there is raised the Perpendicular EA to its Plan, (*d*) this will pass thro' the Center H, of the Sphere, and therefore the same HE, drawn from the Center of the Sphere, will be perpendicular to the Plan of the Circle BFDG. Wherefore HE, both ways produced, will fall in the Poles of the Circle; and accordingly C, will be the other Pole of the Circle BFDG. Q. E. D.

(*c*) 9. 3.

(*d*) Cor. 2. *of this.*

THEO. IX. PROP. X.

A right Line drawn thro' the Poles of any Circle in a Sphere, will be perpendicular to the Plan of the Circle; and will pass thro' the Center of the Circle, and of the Sphere.

Fig. 12. IN the Sphere ABCD, let there be a Circle, as BFDG, thro' the Poles A, C, of which is drawn the right Line AC, cutting the Plan of the Circle in E. I say the right Line AC, is perpendicular to the Plan of the Circle, and passes thro' it's Center (that is, E, is it's Center) and also thro' the Center of the Sphere. For any how drawing thro' E, the two right Lines BD, FG, and joining their Extremes by right Lines drawn from the Poles A, C; AB, AG, AF, AD, will be equal, and also CB, CG, CF, CD, from the Definition of a Pole. Therefore the two Triangles ABC, ADC, have two Sides AB, AC, equal to two Sides AD, AC, and the Base BC, equal to the Base DC. (*a*) Wherefore also the Angles BAC, DAC, will be equal. Therefore because the two Triangles ABE, ADE, have the two Sides

(*a*) 8. 1.

Sides AB, AE, equal to the two Sides AD, AE, and the Angles BAE, DAE contained under them equal, as has been proved, also the Angles AED, AEB, (*b*) will be equal, and consequently right ones. In the same manner we demonstrate, that AEG, AEF, are right Angles. Therefore the right Line AE is at right Angles to the Lines BD, FG. (*c*) Wherefore it will be Perpendicular to the Plan of the Circle, drawn thro' the right Lines BD, EG. Which was the thing first proposed. Now because from A, the Pole of the Circle BFDG, the right Line AE, is drawn perpendicular to its Plan, (*d*) AE will fall in its Center. Therefore E, is the Center of the Circle BFDG. Again, because from E, the Center of the Circle BFGD, is drawn the Perpendicular EA, to its Plan, this (*e*) will also pass thro' the Center of the Sphere. VVherefore the right Line AC is Perpendicular to the Plan of the Circle BFDG, and passes thro' its Center, and the Center of the Sphere. Q E. D.

(*b*) 4. 1.

(*c*) 4. 11.

(*d*) 9. *of this.*

(*e*) Cor. 2. *of this.*

SCHOLIUM.

There are added here these two other Theorems.

I.

If there be a Circle, in a Sphere, and from one of its Poles a right Line be drawn thro' the Center of the Sphere; this Line will be perpendicular to the Plan of the Circle, and produced, will fall in its Center, and the other Pole.

In the Sphere ABCD, whose Center is E, let there be the Circle BGDH, from whose Pole A, thro' E, the Center of the Sphere, is drawn the right Line AE, cutting the Plan of the Circle in F, and the Superficies of the Sphere, in C. I say AE, is perpendicular to the Plan of the Circle, and passes thro' its Center and the other Pole; that is, F is the Center, and C, the other Pole For having drawn the two right Lines BD,GH, any how, and drawn Lines to their Extremes, from the Points A, E; AB, AH, AD, AG, from the Definition of a Pole, will be equal; as also EB, EH, ED, EG, the Fig. 13.

Semi

Semidiameters of the Sphere. Therefore because the two Triangles ABE, ADE, have two Sides AB, AE, equal to two Sides AD, AE, and the Base
(a) 8. 1. *EB equal to the Base ED; (a) the Angles BAE, DAE. will be equal. Therefore the two Triangles ABF, ADF, have two Sides, AB, AF, equal to two Sides, AD, AF; and the Angles BAF, DAF, contain'd under them, e-*
(b) 4. 1. *qual, as just now was shewn. (b) Wherefore the Angles, AFB, AFD, will be equal, and therefore right ones. We demonstrate, in the same manner, that the Angles AFH, AFG, are right ones. Therefore the right Line AF, is at right Angles to the two right Lines BD, GH;*
(c) 4. 11. *(c) wherefore it will be perpendicular to the Plan of the Circle BGDH, drawn thro' the right Lines BD,*
(d) 9. *of this.* *GH: (d) And therefore produced, will fall in the Center of the Circle and the other Pole; and accordingly F, will be the Center of the Circle, and C the other Pole.* Q. E. D.

COROLLARY.

Hence, a great Circle passing thro' one of the Poles of any Circle in a Sphere, passes also thro' the other Pole. For if from one Pole, thro' the Center of the Sphere, be drawn the Diameter of a great Circle, passing thro' that Pole, this will fall in the other Pole, as has been demonstrated. Therefore the same great Circle will pass thro' the other Pole. And because the Diameter of a great Circle, is also the Diameter of the Sphere, it is manifest, that the two Poles of any Circle in a Sphere, are diametrically opposite; and therefore between them there is interposed a Semicircle of a great Circle.

II.

If there is a Circle in a Sphere, and from the Center of the Sphere a right Line be drawn, thro' the Center of the Circle; the said Line will fall in both the Poles of the Circle.

In the last Figure draw thro' E, the Center of the Sphere, and F the Center of the Circle BGDH, the right Line EF, which produce both ways. I say EF, falls in each Pole of the Circle BGDH: For because the right Line

EF,

EF, connecting the Center of the Sphere. and the Center of the Circle BGDH, (e) is perpendicular to the Plan of the same Circle, (f) the same EF, each way produced, will fall in both the Poles of the Circle. Q. E. D. (e) 7. of this. (f) 8. of this.

COROLLARY.

From the whole, it is manifeſt, that theſe four Points, in a Sphere, namely the two Poles of any Circle, its Center, and the Center of the Sphere, are always in one right Line, *viz.* the Diameter of the Sphere; which Diameter it perpendicular to the Plan of the Circle: So that a right Line drawn thro' any two of thoſe Points, will alſo paſs thro' the other two, and be perpendicular to the Plan of the Circle: Likewiſe a right Line drawn thro' one of thoſe Points, perpendicular to the Plan of the Circle, will alſo paſs thro' the other three Points.

THEO. X. PROP. XI.

Great Circles in a Sphere, mutually cut each other in half.

IN the Sphere ABCD, let the two great Circles AC, BD mutually cut each other in the Points E, F. I ſay they mutually biſect each other. (*a*) For becauſe great Circles in a Sphere paſs thro' its Center, the Circles AC, BD, will paſs thro' the Center of the Sphere, which let be G. (*b*) And becauſe the Center of the Sphere is the ſame, with the Center of a Circle paſſing thro' the Center of the Sphere, the Point G, which is put for the Center of the Sphere, will be alſo the Center of both the Circles AC, BD, ſo that it will be in the Plans of both the Circles AC, BD. Alſo the Points E, F, are in each Plan. Therefore three Points E, G, F, are in both the Plans of the Circles AC, BD; and conſequently they will be in their common Section, becauſe only their common Section is in each Plan. (*c*) But their common Section is a right Line. Therefore Fig. 14. (a) 6. of this. (b) Cor. 1. of this. (c) 3. 11.

three Points E, G, F, are in a right Line drawn from E thro' G to F, which because it passes thro' G, the Center of both Circles, and of the Sphere, as has been prov'd, it will be the Diameter of both Circles, and of the Sphere. And therefore it will cut each of them in half, so that EAF, FCE, EBF, FDE, are Semicircles. Q. E. D.

THEO. XI. PROP. XII.

Circles in a Sphere, mutually cutting one another in half, are great ones.

Fig. 15. IN the Sphere ABCD, let the Circles AE, BD, mutually bisect each other in the Points E, F. I say the Circles AC, BD, are great ones. For because they mutually bisect each other, in E, F, the right Line EF, (being drawn) will be the Diameter of them both, since only a Diameter bisects any Circle; and accordingly the right Line EF, being bisected in G, G will be the Center of both the Circles: Which I say also is the Center of the Sphere, and consequently both Circles pass thro' the Center of the Sphere. For if G, be denied to be the Center of the Sphere, and accordingly the Circles AC, BD, are not drawn thro' the Center of the Sphere; we thus demonstrated that G, is the Center, and therefore each Circle passes thro' the Center of the Sphere. *(a)* For raise from G, to the Plan of the Circle AC, the perpendicular GH: Also raise GI, perpendicular to the Plan of the Circle BD. Therefore because the Circles AC, BD, are denied to pass thro' the Center of the Sphere, both the perpendiculars GH, GI, *(b)* will pass thro' the Center. Wherefore the Point G, in which they meet, will be the Center of the Sphere, for othetwise the Center will not be in both: And accordingly both the Circles pass thro' the Center of the Sphere. *(c)* Therefore the Circles AC, BD, passing thro' the Center of the Sphere are great ones. And consequently Circles in a Sphere mutually bisecting each other, are great ones. Q. E. D.

(a). 12. 11.

(b) Cor. 2 *of this.*

(c) 6. *of this.*

SCHO-

SCHOLIUM.

Here you ſee an admirable way of arguing. For from the Denial of G's being the Center of the Sphere, it is demonſtrated in the Affirmative that G is the Center of the Sphere. Which manner of arguing alſo is uſed by Euclid, *in* Prop. 12. Lib. 9, *and by* Cardan in Lib. 5. Prop. 201, *as we have mentioned in the Scholium of the ſame Propoſition.*

THEO. XII. PROP. XIII.

If a great Circle in a Sphere cuts any other Circle at right Angles; it will alſo cut it in half, and paſs thro' its Poles.

LET the great Circle ABCD in a Sphere cut the Circle BED, at right Angles, in the Points B,D, that is let the Plan of the Circle ABCD, be at right Angles, to the Plan of the Circle BED, and let their common Section be the right Line BD. I ſay the Circle ABCD, cuts the Circle BED, in half, and paſſes thro' its Poles. (*a*) For the Center F, of the great Circle ABCD, being found, which alſo will be the Center of the Sphere: (*b*) For when a great Circle is drawn thro' the Center of the Sphere, (*c*) its Center, will be the ſame as the Center of the Sphere.) (*d*) Draw the perpendicular FG, from F to the Plan of the Circle BED, (*e*) which will fall in the common Section BD. And let it fall in G. Then becauſe it likewiſe falls in the Center of the Circle BED, G will be the Center of the Circle BED (*f*) and therefore BD drawn thro' G, will be a Diameter of the ſame: And becauſe it divides the Circle BED in half, alſo the great Circle ABCD, drawn thro' the right Line BD, will divide it in half. Which was the firſt thing propoſed. Now becauſe the right Line FG, is in the Plan of the Circle ABCD, that produced, will fall to the Points A, C, which are in the Superficies of the Sphere: (*g*) It will likewiſe fall in each Pole of the Circle

Fig. 16.
(*a*) 1. 1.
(*b*) 6. *of this.*
(*c*) Cor. 1. *of this.*
(*d*) 11. 11.
(*e*) 38. 11.
(*f*) Cor. 1. *of this.*
(*g*) 8. *of this.*

cle BED, because it is drawn from F, the Center of the Sphere, perpendicular to the Plan of the Circle. Therefore A, C, are the Poles of the Circle BED ; and according the great Circle ABCD, passes thro' the Poles of the Circle BED. Which was the second Thing proposed to be demonstrated.

SCHOLIUM.

This, together with the 8th, 9th, and 10*th. Propositions, and their Scholium, take place, when the Circle, BD, is a great Circle, and passes thro' the Center of the Sphere. For it is manifest, the Demonstration is nighly the same.*

THEO. XIII. PROP. XIV.

If a great Circle in a Sphere bisects another Circle, which is not a great one ; it will cut that other Circle at right Angles, and pass thro' its Poles.

Fig 17. LET the great Circle ABCD, in a Sphere, cut the lesser Circle BED, in half, in the Points B, D, and let their common Section be the right Line BD. I say the Circle ABCD, cuts the Circle BED, at right Angles, and passes thro' its Poles. For because the Circle BED, is bisected in B, D, that is, in Semicircles, the common Section BE, will be its Diameter. Therefore BD, being bisected in F, F will be the Center of the Circle BED. (*a*) And assuming G, the Center of the Sphere, which also will be the Center of the great Circle ABCD, draw from G to F, the right Line GF, (*b*) which will be perpendicular to the Plan of the Circle BED : (*c*) And so the Plan of the great Circle ABCD, drawn thro' the right Line FG, will be at right Angles to the Plan of the Circle BED. Therefore the great Circle ABCD, cuts the lesser Circle BED, at right Angles: Which was the first thing to be demonstrated. And because it has been shewn, that the right Line FG, drawn from G, the Center of the Sphere, to the Plan of the Circle BED, is perpendicular, FG, each way produced, (*d*)

(*a*) 2. *of this.*
(*b*) 7. *of this*
(*c*) 18. 11.

(*d*) will fall in the Poles of the Circle BED. Wherefore because GF existing in the Plan of the Circle ABCD, produced falls in its Circumference in the Points A, C, which also are in the Superficies of the Sphere; A, C, will be the Poles of the Circle BED; and therefore the great Circle ABCD, passes thro' the Poles A, C, of the lesser Circle BED. Which was the second thing proposed. (*d*) 8. *of this.*

THEO. XIV. PROP. XV.

If a great Circle in a Sphere passes thro' the Poles of another Circle, it will bisect this other Circle, and cut it at right Angles.

LET the great Circle ABCD, in a Sphere, pass thro' the Poles A, C, of the Circle BED: I say the Circle ABCD cuts the Circle BED, in half, and at right Angles. For from one Pole to the other draw the right Line AC, cutting the Plan of the Circle BED in F. (*a*) Then because the right Line AC, is perpendicular to the Plan of the Circle BED, and passes thro' the Center of the Sphere, and the Center of the Circle BED; F, will be the Center of the Circle BED. Therefore since the great Circle ABCD, cutting the Circle BED, passes thro the right Line AC, and so thro' the Center F, the common Section BFD, will be a Diameter of the Circle BED. Therefore the Circle BED is bisected; I say also and at right Angles. For because the right Line AC, has been shewn to be perpendicular to the Plan of the Circle BED, also the Plan of the great Circle ABCD, drawn thro' the right Line AC, (*b*) will be at right Angles to the Plan of the Circle BED. Q. E. D. Fig. 18. (*a*) 10. *of this.* (*b*) 18. 11.

SCHOLI-

SCHOLIUM.

There are added Four other Theorems, in this Order, in another Version.

I.

If a great Circle in a Sphere, passes thro' the Poles of any other great Circle, this shall mutually pass thro' the Poles of that.

Fig. 19. *Let the great Circle ABCD, in a Sphere, pass thro' the Poles A, C, of the great Circle BD. I say the great Circle BD, will also pass thro' the Poles of the great Circle ABCD. For because the great Circle ABCD, passes thro' the Poles of the Circle BD, it (a) will cut it at right Angles. Wherefore reciprocally the great Circle BD, will cut the Circle ABCD, at right Angles; (b) and therefore it will pass thro' its Poles. Which was proposed.*

(a) 15. of this.
(b) 13. of this.

II.

If a Circle in a Sphere, passes thro' the Poles of another Circle, it will be a great Circle, bysecting that other Circle, and also at right Angles to it.

Fig. 20. *Let the Circle ABCD in a Sphere, pass thro' the Poles A, C, of the Circle BD. I say it is a great Circle, and cuts the Circle BD in half, and at right Angles. For joyn the Poles A, C, by the right Line AC, which necessarily, will be in the Plan of the Circle ABCD, because its Circumference, is supposed to pass thro' the same Poles A, C. But because the right Line AC, drawn thro' the Poles A, C, of the Circle BD, (a) passes thro' the Center of the Sphere, also the Circle ABCD, (because it is drawn thro' the right Line AC.) will pass thro' the Center of the Sphere; (b) and consequently will be a great Circle. Wherefore since it is supposed to pass thro' the Poles A, C, of the Circle BD, (c) it will cut it in half, and at right Angles. VVhich was proposed.*

(a) 10. of this.
(b) 6. of this.
(c) 15. of this.

III.

III.

If a Circle in a Sphere cuts another Circle in half, and also at right Angles; it will be a great Circle, and passes thro' the other Circle's Poles.

Let the Circle ABCD, in a Sphere, cut the Circle BD, in half, and at right Angles. I say it is a great Circle, and passes thro' the Poles of the Circle BD. For let the right Line BD be their common Section. Therefore because the Circle ABCD, cuts the Circle BD, in half, the right Line BD, to wit, their common Section, will be the Diameter of the Circle BD, and therefore bisects the right Line BD, in E: Whence E, will be the Center of the Circle. Now draw in the Plan of the Circle ABCD, the right Line EA, perpendicular to BD. Then because the Circle ABCD, cuts the Circle BD at right Angles, EA, (from Def. 4. Lib. 11. Euclid) *will be at right Angles, to the Plan of the Circle BD; and accordingly because it is drawn from E, its Center, it will (d) fall in both the Poles: It also falls in the Circumference of the Circle ABCD, existing in the Superficies of the Sphere, at the Points A, C. Therefore A, C, are the Poles of the Circle BD; and so the Circle ABCD, passes thro' the Poles A, C, of the Circle CD. Wherefore from the precedent Theorem, it will be a great Circle. But it has been prov'd that it passes thro' the Poles of the Circle BD. Therefore what was propos'd, is manifest.* Fig. 21.

(*d*) Scol.8. *of this.*

IV.

If there is a Circle in a Sphere, and from one of the Poles be drawn to its Plan a perpendicular Line equal to its Semidiameter; the said Circle will be a great one.

Let there be a Circle, as AB in a Sphere, from the Pole C of which, to its Plan, is let fall the Perpendicular CD, equal to its Semidiameter. I say AB is a great Circle. For because CD, is perpendicular to the Circle AB, it (h) will fall in the Center of the Circle, and produced will fall in the other Pole, which let be E. Fig. 22.

(*h*) 9. *of this.*

VVhence

(*i*) Cor. 2. *of this* *VVhence D, is the Center of the Circle AB: And (i) therefore the Perpendicular CD, will paſs thro' the Center of the Sphere. Now draw thro' the right Line*
(*k*) 1. *of this.* *CE, in the Sphere, a Plan any how (k) making the Circle AEBC, which becauſe it paſſes thro' the Center of*
(*l*) 6. *of this.* *the Sphere, (l) will be a great Circle. VVhich Circle AB, cut, in the Points A, B, and draw the Semidiameter DB which, from the Hypotheſis is equal to CD. But becauſe CD is Perpendicular to the Circle AB, the Angle CDB, will be (from* Def. 3. Lib. 11. Euclid.)
(*m*) Schol. 13. 6. *a right one. (m) VVherefore BD is a mean Proportional between CD, DE, that is, as CD, to DB; ſo will BD be to DE. But CD is equal to BD. And therefore DE, will be equal to the ſame BD; and conſequently CD, DE will be equal, between themſelves. Therefore becauſe CE, has been proved to paſs thro' the Center of the Sphere, D will be the Center of the Sphere. But it was alſo the Center of the Circle AB. Therefore the Center of the Sphere, and the Center of*
(*n*) 6. *of this.* *the Circle AB, is the ſame; (n) whence accordingly the Circle AB is a great one. VVhich was propoſed.*

THEO. XV. PROP. XVI.

If there is a great Circle in a Sphere, a right Line drawn from one of the Poles to its Circumference, is equal to the ſide of Square inſcribed in a great Circle.

Fig. 23. LET there be a great Circle AB, in a Sphere, from whoſe Pole C, to its Circumference, is drawn the right Line CB. I ſay CB is equal to the Side of the Square inſcribed in the Circle AB, or any other great one.
(*a*) 11. 11. For (*a*) draw from C, to the Circle AB, the Perpendi-
(*b*) 9. *of this.* cular CE, (*b*) which will fall in its Center, which let be E, and produced will fall in the other Pole, which let be D. Now let there be drawn thro' the right Lines AB, CD, a Plan, (*c*) making the Circle ADBC in the
(*c*) 1. *of this.* Sphere; which becauſe it paſſes thro' E the Center of the

Sphere

Sphere (for E, the Center of the great Circle AB, which passes thro' the Center of the Sphere, (*d*) will be the same, as the Center of the Sphere) (*e*) will be a great Circle; and therefore it will (*f*) bisect the great Circle AB. VVhich likewise from hence is manifest, because it passes thro' its Poles. (*g*) For from hence it is that it bisects it. Let therefore the common Section BEA be the Diameter. And because CE, is drawn perpendicular to the Circle AB, it will be perpendicular (from *Def.* 3. Lib. 11. *Euclid*) to the right Line AB. Therefore two Diameters AB, CD, in the great Circle ADBC, mutually cut each other at right Angles: (*h*) and accordingly, as is demonstrated in the fourth Book of *Euclid*, CB, is the Side of a Square inscribed in the great Circle ADBC, and likewise in the great Circle AB. *Q. E. D.*

(*d*) Cor. 1. *of this.*
(*e*) 6. *of this.*
(*f*) 11. *of this.*
(*g*) 15. *of this.*
(*h*) 6. 4.

COROLLARY.

But because the four right Angles, at the Center E, are equal, and (*i*) consequently the four Arc's BC, CA, AD, DB, which they comprehend, equal, *viz.* Quadrants, it is manifest, that the Pole of a great Circle, in a Sphere, is distant from its Circumference, a Quadrant of a great Circle. For C, the Pole of the great Circle AB, is distant from its Circumference, by the Quadrant CB, and there is the same reason for the others. For (*k*) always a right Line drawn from the Circumference of a great Circle to its Pole, is equal to the Side of a Square inscrib'd in a great Circle, and therefore it subtends a Quadrant in a great Circle.

(*i*) 26. 3.
(*k*) 16. *of this.*

SCHOLIUM.

The Converse of this is likewise demonstrated, in the other Version, in this Theorem.

If there is a Circle in a Sphere, and a right Line be drawn from its Poles to its Circumference, equal to the Side of a Square inscrib'd in it, that Circle will be a great one.

In the last Figure, let there be drawn the right Line CB, from the Pole C, of the Circle AB to its Circumference, equal to the Side of the Square inscribed in
(l) 11. 11. *the Circle AB. I say AB is a great Circle. For (l) let there be drawn from C to the Circle AB, the Per-*
(m) 9. of this. *pendicular CE, which (m) will fall in its Center, which let be E. And having drawn the Semidiameter EB, the Angle E (from* Def. 3. lib. 11. Euclid*) will be a*
(n) 47. 1. *right one. (n) Therefore the Square of CB, that is, the Square describ'd in the Circle AB, is equal to the Squares of BE, CE: But the Square of the Semidiameter BE, is half the Square describ'd in the Circle AB, as presently shall be demonstrated. And therefore the Square of CE, will also be half of the Square describ'd in the same Circle; whence the Squares of BE, CE, will be equal to each other, and consequently the Lines BE, CE. Wherefore because CE is drawn from the Pole C, of the Circle AB, perpendicular to its*
(o) Schol. 15. *of this.* *Plan, and it has been proved to be equal to the Semiter BE, (o) AB will be a great Circle.*

LEMMA.

In any Circle the Square of the Semidiameter is half of the Square inscrib'd in it.

Fig. 24. *In the Circle, whose Center is E, let there be drawn the Diameters AC, BD, crossing each other at right Angles, in the Center E. Therefore the right Lines AB, BC, CD, DA, being drawn ABCD will be a Square, inscrib'd in the Circle, as is manifest from* Prop. 6. lib. 4. Euclid. *But because the Squares of the equal Semi-*
(p) 47, 1. *diameters EA, EB, are equal between themselves, (g) they both together are equal to the Square of AB; wherefore the Square of EA, will be half the Square of AB. VVhich was proposed.*

THEO-

THEO. XVI. PROP. XVII.

If there be a Circle in a Sphere, from whose Pole to its Circumference is drawn a right Line equal to the Side of a Square inscrib'd in a great Circle, the aforesaid Circle will be a great one.

LET there be a Circle, as AB, in a Sphere, from whose Pole C to its Circumference is drawn the right Line CA, equal to the Side of a Sphere inscrib'd in a great Circle of the Sphere. I say AB is a great Circle. For draw a Plan thro' the right Line AC, and the Center of the Sphere, (*a*) making the Circle ACB in the Sphere, which (*b*) will be a great one, because it's drawn thro' the Center of the Sphere. Draw also from C, the right Line CB to the Point B, in which the great Circle ACB, cuts the Circle AB; then from the *Def.* of a Pole, the right Line CB, will be equal to the right Line CA. Therefore because AC, is the Side of a Square inscrib'd in the great Circle ACB, CB will be also the Side of the same Square; and therefore the two Arc's AC, CB, will be Quadrants, making up the Semicircle ACB, because the four equal Sides of the Squares, (*c*) subtend four equal Arc's of the Circle. Therefore the right Line AB, the common Section of the Circles, will be a Diameter of the great Circle ACB; and accordingly of the Sphere. But because the great Circle ABC passing thro' the Poles of the Circle AB, (*d*) cuts it in half, the common Section AB, will also be a Diameter of the Circle AB; and accordingly, since it is likewise the Sphere's Diameter, AB will be a great Circle. Q. E. D.

Fig. 25.

(*a*) 1. *of this.*

(*b*) 6. *of this.*

(*c*) 28. 3.

(*d*) 15. *of this.*

PROB. II. PROP. XVIII.

To draw a right Line equal to the Diameter of any Circle in a given Sphere.

Fig. 26. 27. LET any Circle ABCD be given in a Sphere: It is required to find its Diameter. Having assumed any where three Points, A, B, D, in the Circumference of the Circle, and drawn the right Lines AB, AD,
(*a*) Schol. 22. 1. BD, (*a*) make the Triangle EFG equal to the Triangle ABD, so that the Side EF be equal to the Side AB, EG, to AD, and FG to BD. For the three Intervals AB, AD, BD taken in the Superficies of the Sphere may by help of a pair of Compasses be transferr'd on a Plan; and so a Triangle may be constituted, whose three Sides are equal to those three Distances. Again from G, F, draw the Perpendiculars FH, GH, to the right Lines EF, EG, concurring in H, and joyn the Points E, H. I say EH, is equal to the Diameter of the Circle ABCD. For having drawn the Diameter AC, joyn the Points D, C.
(*b*) Schol. 32. 1. Now (*b*) because the four Angles of the quadrilateral Figure EFHG, are equal to two right ones, and EFH, EGH, are right Angles, also FEG, FHG, will be equal to two right ones; and therefore in the quadrilateral Figure EFHG, any two opposite Angles, are equal to
(*c*) Schol. 22. 3. two right Angles. (*c*) Wherefore a Circle may be describ'd about it: VVhich being describ'd, the Angles EFG, EHG, in the same Segment, whose Chord is EG,
(*d*) 27. 3. (*d*) will be equal. (*e*) But the Angle EFG, is equal to
(*e*) 8. 1. the Angle ABD; since the two Sides EF, FG, are equal to two Sides AB, BD, and the Base EG, to the Base
(*f*) 27. 3. AD, from Construction, (*f*) and also the Angle ABD, equal to the Angle ACD. Therefore also the
(*g*) 31. 3. Angle EHG, will be equal to the Angle ADC, (*g*) which here likewise is a right Angle, being in the Semicircle ADC. VVherefore the Triangles EHG, ACD, have two Angles equal to two Angles, and also the Side EG, subtending one of the equal Angles equal to the Side AD. (*h*) VVherefore also the Side EH, will be
(*h*) 26. 1. equal to the Side AC. Therefore we have drawn the right

right Line EH, equal to the Diameter AC, of the Circle ABCD. Q. E. F.

PROB. III. PROP. XIX.

To draw a right Line equal to the Diameter of a given Sphere.

HAving assumed the two Points A, B, any where on the given Sphere, describe from the Pole A, and with the distance AB, the Circle BD, to (*a*) whose Diameter make the right Line FG, equal, (*b*) and make upon FG, the Triangle EFG, having each of the other Sides EF, EG, equal to the drawn Line AB, *viz.* in assuming with a pair of Compasses the interval AB, &c. Again draw from F, G, the Perpendiculars FH, GH, to the Lines EF, EG, meeting in H; and joyn the Points E, H. I say EH, is equal to the Diameter of the given Sphere. For having drawn the Diameter AC of the Sphere, draw a Plan, thro' the right Lines AB, AC, (*c*) making the Circle ABCD, (*d*) which will be a great one, because it is drawn thro' the Diameter of the Sphere, and so thro' the Center of the same. Wherefore the same drawn thro' A, the Pole of the Circle BD (*e*) will bisect the Circle BD; and accordingly the common Section BD, will be a Diameter of the Circle BD: And drawing the right Lines AD, DC, the two Sides AB, DB, will be equal to the two Sides EF, FG, as also the Bases AD, EG. For FG, is equal from Construction, to the Diameter BD: And both EF, EG, to AB, or AD. (*f*) Therefore also the Angles ABD, EFG, will be equal. (*g*) But the Angle ACD, is equal to the Angle ABD: And also the Angle EHG, to the Angle EFG, as has been demonstrated in the precedent Proposition. Therefore likewise the Angles ACD, EHG, will be equal. Also the right Angles ADC, EGH, are equal, and likewise the Sides AD, EG. (*h*) Therefore the right Line EH, will be equal to the right Line AC. Wherefore we have drawn the right Line EH, equal to the Diameter AC, of the given Sphere. Q. E. F.

Fig. 26. 27.
(*a*) 18. *of this.*
(*b*) Schol. 22. 1.
(*c*) 1. *of this.*
(*d*) 6. *of this.*
(*e*) 15. *of this.*
(*f*) 8. 1.
(*g*) 27. 3.
(*h*) 26. 1.

SCHOLIUM.

The following Theorem is added in the other Version.

A right Line drawn from the Pole of any Circle in a Sphere, to its Superficies, equal to a right Line drawn from the same Pole, to the Circumference of the Circle, falls in the Circumference of the said Circle.

Fig. 28. *Let there be any how drawn the right Line AD, from the Pole A of the Circle BC, in a Sphere, to its Circumference, which will be lesser than the Diameter of the Sphere, and therefore lesser than the Diameter of a great Circle in the Sphere (because the Diameter of a Sphere is the greatest of all right Lines drawn in a Sphere.) Now draw from the same Pole A, to the Superficies, the right Line AE, equal to AD. I say the right Line AE, falls in the Circumference of the Circle BC. For if it does not, thro' the right Line AE, and the Center of the Sphere, draw a Plan, (i) making the Circle ABC, in the Sphere, which (k) will be a great one, as being drawn thro' the Center of the Sphere. Likewise let the Circle ABC, cut the Circle BC, in the Points B, C. Therefore the right Line AE, will not fall in the Points B, C; because it is supposed not to fall in the Circumference of the Circle BC. Whence the right Line AB being drawn, this will be, from the* Definition *of a Pole, equal to AD, and therefore to the right Line AE. And because both AB, AE, are lesser than the Diameters of the great Circle ABC, as has been said, (l) the Arc's AB, AE, because they are Segments lesser than a Semicircle, will be equal,* viz. *the Part to the Whole: which is absurd. Therefore the right Line AE, falls in the Circumference of the Circle BC, which was proposed.*

(i) 1. *of this.*
(k) 6. *of this.*
(l) 28. 3.

THEO-

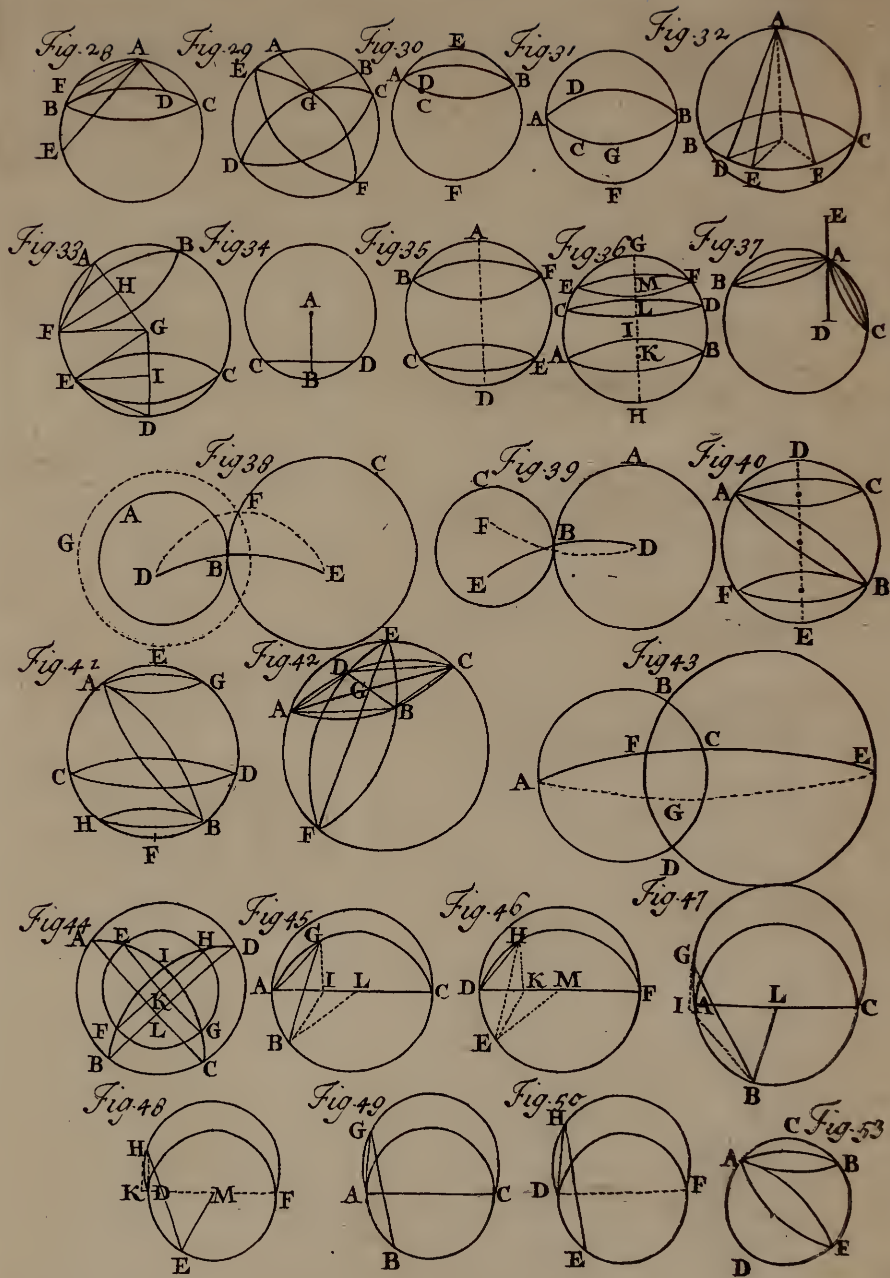


Fig. 28
Fig. 29
Fig. 30
Fig. 31
Fig. 32
Fig. 33
Fig. 34
Fig. 35
Fig. 36
Fig. 37
Fig. 38
Fig. 39
Fig. 40
Fig. 41
Fig. 42
Fig. 43
Fig. 44
Fig. 45
Fig. 46
Fig. 47
Fig. 48
Fig. 49
Fig. 50
Fig. 53

PROB. IV. PROP. XX.

To describe a great Circle through two Points given, in the Superficies of a Sphere.

LET there be given the two Points A, B, in a spherical Superficies, thro' which a great Circle is required to be drawn. Now if the Points A, B, are diametrically opposite, it is certain that an infinite number of great Circles may be described thro' them, *viz.* in drawing an infinite number of Plans thro' the Diameter connecting these two Points. But if the Points A, B, are not in the Diameter of the Sphere, describe the Circle CD, from the Pole A, and with a Distance equal to the Side of a Square inscribed in a great Circle, (*a*) which will be a great Circle, since the right Line drawn from the Pole A, to its Circumference, is equal to the Side of the inscribed Square in a great Circle, and because of the Interval, by which the Circle CD is described. This Interval is thus found. The Diameter of the Sphere being found, as in the preceding *Prop.* the Side of the Square inscribed in a Circle described with that Diameter, will be the Interval sought. Likewise from the Pole B, with the same Interval, describe the Circle EF, (*b*) which will also be a great Circle. Let this cut the first in the Point G, from which draw the right Lines GA, GB; each of which from Construction, will be equal to the Side of an inscribed Square in a great Circle. For with such an Interval are the Circles CD, EF, described. Therefore GA, GB, are equal. Now from the Pole G, and with the Interval GA, let there be described the Circle AEDFCB, (*c*) which will be a great one. But because the right Line GB, is equal to GA, drawn to the Superficies of the Sphere, (*d*) it will fall in the Circumference of the Circle AEDFCB. And accordingly the describ'd Circle AEDFCB, will be a great one passing thro' the two given Points A, B, in the Superficies of the Sphere. Q. E. D.

Fig. 29.

(*a*) 17. *of this.*

(*b*) 17. *of this.*

(*c*) 17. *of this.*

(*d*) Schol. 19. *of this.*

PROB-

PROB. V. PROP. XXI.

To find the Pole of any given Circle in a Sphere.

Fig. 30. 31. LET the Pole of the given Circle AB, be required, which, first, let not be a great one. Having assumed the two Points C, D, any where in the Circumference, (a) divide the Arc's CAD, CBD, in half, in A, B, (b) thro' which let there be describ'd the great Circle AEB; whose Arc AEB bisect in the Point E. I say E, is the Pole of the Circle AB; for because the Arc's AC, AD, are equal, as also BC, BD, the whole Arc's ACB, ADB, will be equal. Wherefore because the great Circle AEB, bisects the Circle AB, which is not a great one, in the Points A, B, (c) it will pass thro' its Poles. Therefore the Point E, equally distant from the Circumference of the Circle AB, is the Pole of the Circle AB. In the same manner, if the other Arc AFB, is bisected in F, F will be the other Pole of the Circle AB.

(a) 30. 3. (b) 20. *of this.* (c) 14. *of this.*

But now, let the given Circle AB, be a great one. Having again any how assum'd the Points C,D, (d) and bisected the Arc's CAD, CBD, in A, B, we prove that the Arc's ACB, ADB, are equal; and accordingly both of them are equal to a Semicircle of a great Circle. Therefore dividing one of the Semicircles, *viz.* ACB, in half in G, a right Line GA subtending a Quadrant, will be the side of a Square inscrib'd in the great Circle AB; as is manifest from *Prop.* 6. *lib.* 4. *Euclid.* Therefore, from the Pole G, and with the distance GA describe the Circle AEB, (e) which will be a great one. Lastly, bisect the Arc AEB, in E. I say E is the Pole of the Circle AB. For because the great Circle ACB, passes thro' G, the Pole of the great Circle AEB; (f) AEB will likewise pass thro' the Poles of the Circle ACB. VVherefore the Point E, equally remote from the Circumference of the Circle ACB, is the Pole of the Circle ACB. In the same manner, dividing the Arc AFB, in half, in F; F will be the other Pole of the Circle ACB. Q. E. F.

(d) 30. 3. (e) 17. *of this.* (f) Schol. 14. *of this.*

SCHO-

SCHOLIUM.

The following two Theorems are demonstrated in the other Version.

I.

If there be taken any Point, in the Superficies of a Sphere, and from the same to the Circumference of any given Circle in the Sphere there are drawn more than two equal right Lines: The aforesaid assumed Point is the Pole of that Circle.

Let A be the Point assumed in the Superficies of the Sphere ABC, from which to the Circumference of the Circle BC, there fall more than two right Lines, as AD, AE, AF. I say A is the Pole of the Circle BC. (a) *For draw from A, to the Plan of the Circle BC, the Perpendicular AG, and joyning the right Lines DG, EG, FG; then, from* Def. 3. lib. 11. Euclid, *all the three Angles at G, will be right ones.* (b) *Wherefore the Square of AD is equal to the Squares of AG, GD; the Square of AE, to the Squares of AG, GE, and* &c. *Therefore because the Squares of the equal right Lines AD, AE, AF, are equal; also the Squares of AG, GD, together will be equal to the Squares of AG, GE together, as also to the Squares of AG, GF, together; Therefore taking away the common Square of the right Line AG, the remaining Squares of the right Lines GD, GE, GF, and consequently also the said Lines, will be equal.* (c) *Therefore G will be the Center of the Circle BC;* (d) *and accordinly the right Line GA, drawn from the Center G, perpendicular to the Circle BC, falls in the Pole of that Circle. Therefore the Point A, is the Pole of the Circle BC. Which was proposed.*

Fig. 32.
(a) 11. 11.
(b) 47. 1.
(c) 9. 3.
(d) Schol. *this.*

II.

Circles in a Sphere, from whose Poles to their Circumferences are drawn equal right Lines, are equal. And right Lines drawn from the

Poles of equal Circles, to their Circumferences, are equal.

Fig. 33. *In the Sphere ABCDEF, let there be two Circles, as BF, CE, from whose Poles A, D, the right Lines AF, DF, drawn to their Circumferences, are equal. I say the Circles BF, CE, are equal.* (a) *For let there be drawn the Perpendiculars AH, DI, from the Poles A, D, to the Plans of the Circles,* (b) *which will fall in their Centers, H, I, and from thence produced, in the other Poles,* (c) *and so in G, the Center of the Sphere. Therefore having drawn the Semidiameters FG, EG, of the Sphere, and the Semidiameters FH, EI, of the Circles; because the Sides AG, GF, are equal to the Sides DG, GE, and the Base AF, to the Base DE, the Angles AGF, DGE,* (d) *will be equal. But the Angles H, I, from* Def. 3. lib. 11. Euclid. *Are right ones. Therefore the Triangles FGH, EGI, have two Angles equal to two Angles: Also the side FG is equal to the Side EG:* (e) *Therefore also the Semidiameters FH, EI, will be equal; and consequently the Circles BF, CE are equal. Which was the thing first proposed.*

(a) 11. 11.
(b) 9. of this.
(c) 10. of this.
(d) 8. 1.
(e) 26. 1.

Now let the Circles BF, CE, be equal. I say the Lines AF, DE, drawn from the Poles to their Circumferences are equal. For the same things being constructed, the Semidiameters FH, EI, will be equal, (f) *and the Circles, equally distant from the Center of the Sphere. Wherefore the Perpendiculars GH, GI, will be equal; and consequently the Lines AH, DI, will be equal. Therefore because the Sides AH, HF, are equal to the Sides DI, IE, and contain the equal Angles at H, I, as being right ones, from* Def. 3. lib. 11. Euclid, (g) *the Bases AF, DE, will be equal. Which was the second thing proposed.*

(f) 6. of this.
(g) 4. 1.

THEO-

THEO. XVII. PROP. XXII.

If a right Line drawn thro' the Center of a Sphere, cuts another Line not drawn thro' the Center, in half, it will be at right Angles to it. And if it cuts it at right Angles, it also bisects it.

LET the right Line AB, drawn thro' the Center A, of a Sphere, bisect the Line CD, not drawn thro' the Center, in the Point B. I say it cuts CD at right Angles. For a Plan being drawn thro' the right Lines AB, CD, *(a)* making the Circle CD, *(b)* (which will be a great one, because it passes thro' the Center of the Sphere,) because the right Line AB, in the Circle CD, passing thro' its Center A, bisects the right Line CD, not passing thro' the Center, in B, (*c*) it will cut it at right Angles. And if it cuts it at right Angles, it will bisect it. Q. E. D. Fig. 34. *(a)* 1. *of this.* *(b)* 6. *of this.* *(c)* 3. 3.

SCHOLIUM.

There is here added in the Greek Version another Theorem, which is altogether the same, as is demonstrated in the 7th. Prop. *Therefore it is needless here to repeat it.*

End of the first BOOK

THE

The Spherical Elements

OF

THEODOSIUS.

BOOK II.

DEFINITION.

CIRCLES in a Sphere are said to mutually touch one another, when the common Section of their Plans touches each Circle.

For because a right Line touching any Circle in a Sphere, likewise touches the Superficies of the Sphere in the same Point in which it touches the Circle (for if it did not touch it, but cut it, it would also necessarily cut the Circle, because it is in its Plan, and connects two Points in the Superficies of the Sphere, viz. *in which it is said to cut it; which two Points also are in the Circumference of the Circle; since the Plan of the Circle is drawn thro' that Line, and accordingly is cut*

cut by it in those two Points.) From thence it is that the Circumferences of two Circles, the common Section of which (to wit, which their Plans produced make) touches each Circle, have only that Point in which it touches the Sphere, common: Because in that Point, and no other, the aforesaid common Section can touch both Circles; since that all the other Points of it, are without the Superficies of the Sphere, and so without each Circle. Therefore Theodosius *has rightly defined, that Circles are mutually said to touch one another in a Sphere, when their common Section touches each Circle.*

THEO. I. PROP. I.

Parallel Circles in a Sphere, have the same Poles.

LET there be the Parallel Circles BF, CE, in the Sphere ABCDEF. I say they have the same Poles. (*a*) For let A, D, be the Poles of the Circle BF, and the right Line AD, (*b*) will be perpendicular to the Circle BF, and will pass thro' the Center of the Sphere. Therefore because the right Line AD is perpendicular to the Circle BF, (*c*) it will be also perpendicular to the parallel Circle CE. Whence since it passes thro' the Center of the Sphere, as has been shewn; (*d*) it falls in the Poles of the Circle CE. Therefore A, D, are the Poles of the Circle CE. But they are likewise the Poles of the Circle BF. Q. E. D.

(*a*) 21. 1. *of this.* Fig. 35. (*b*) 10. 1. *of this.* (*c*) Schol. 14. 11. (*d*) 8. 1. *of this.*

THEO. II. PROP. II.

Circles in a Sphere, which have the same Poles, are parallel.

IN the last Figure, let the Circles BF, CE, have the same Poles: Now I say they are parallel. For having drawn the right Line AD, (*a*) this will be perpendicular

(*a*) 10. 1. *of this.*

(b) 14. 11. lar to both the Circles. (b) Wherefore the Plans of the Circles will be parallel. Q. E. D.

SCHOLIUM.

The following Theorem is likewise demonstrated in the other Version.

There are not more than two Circles in a Sphere, Equal, and Parallel.

Fig. 36. *In any Sphere let there be, if possible, more than two Circles, equal, and parallel,* viz. *the three AB, CD, EF* (c) 1. of this. *(c) which will have the same Poles. Therefore let their Poles be G, H, and draw the right Line GH,* (d) 10. 1. of this. *(d) which will pass thro' I, the Center of the Sphere, and thro' K, L, M, the Centers of the Circles, and also will be perpendicular to the Circles AB, CD, EF. Therefore because the Circles AB, CD, EF, are equal, they* (e) 6. 1. of this. *(e) will be equally distant from the Center I, of the Sphere. Whence, by* Def. 6. lib. 1. *of this, the Perpendiculars IK, IL, IM, will be equal, to wit, the Part IL, and the Whole IM : which is absurd.* Q. E. D.

THEO. III. PROP. III.

If two Circles in a Sphere, cut in the same Point, the Circumference of a great Circle, passing thro' their Poles, these Circles will mutually touch one another.

Fig. 37. LET the two Circles AB, AC, cut in the Point A, the Circumference of the great Circle ABC, passing thro' their Poles. I say the Circles AB, AC, mutually touch one another in the Point A. For because the great Circle ABC, passes thro' the Poles of the Circles AB, AC, (a) 15. 1. of this. (a) it will bisect them at right Angles. Therefore the common Sections of the Circle ABC, and the Circles AB, AC, *viz.* the right Lines AB, AC, will

be

be the Diameters of the Circles AB, AC. Let also the common Section of the Plans, in which are the Circles AB, AC, be the right Line DE, which will pass thro' the Point A, because the Plans are supposed to cut the Circle ABC, in A. Now since the Plan of the Circle ABC, has been proved to be at right Angles to the Plans of the Circles AB, AC, the Plans of the Circles AB, AC, will be likewise at right Angles to the Circle ABC; (*b*) and therefore DE, their common Section, will be perpendicular to the Plan of the Circle ABC, whence also it will be perpendicular to the Diameters AB, AC, in the same Plan, from *Def.* 3. *lib.* 11. *Euclid.* (*c*) Wherefore DE, touches both the Circles AB, AC, in A; and accordingly, by the Definition of this Book, the Circles AB, AC, mutually touch one another in the Point A. Q. E. D. (*b*) 19. 11. (*c*) Cor. 16. 3.

THEO. IV. PROP. IV.

If two Circles in a Sphere mutually touch each other, a great Circle drawn thro' their Poles, will pass thro' their Point of Contact.

LET the Circles AB, CB, in a Sphere, mutually touch each other in B; and thro' D, the Pole of the Circle AB, and E, the Pole of the Circle CB, let there be (*a*) describ'd the great Circle DE. I say the Circle DE, passes thro' the Point of Contact B. For if it does not pass thro' B the Point of Contact, let it cut the Circumference, for Example, of the Circle CB, in F. Now from the Pole D, and with the distance DF, describe the Circle FG, which because it is describd with a greater distance, than the Circle AB is, it will cut the Circle CB, in F. But because the two Circles BF, GF, in a Sphere, cut in the same Point F, the great Circle DEF, described thro' their Poles, the two Circles GF, CF, (*b*) will touch one another in F: But they will likewise mutually cut one another in F. Which is absurd. Q. E. D. Fig. 38. (*a*) 20. 1. *of this.* (*b*) 3. *of this.*

THEO-

THEO. V. PROP. V.

If two Circles in a Sphere mutually touch one another, a great Circle describ'd thro' the Poles of one of them, and their Point of Contact, will also pass thro' the Poles of the other Circle.

Fig. 39. LET the two Circles AB, CB, in a Sphere, mutually touch one another in B, and let D, E be their Poles. I say a great Circle describ'd thro' D, the Pole of the Circle AB, and the Point of Contact B, also passes thro' E, the Pole of the Circle CB. For if it can be, let it not pass thro' E, cut thro' some other Point F, and so DBF will be a great Circle. Now having (*a*) describ-ed the great Circle DE, thro' the Poles D, E, (*b*) which will pass thro' B, the Point of Contact, the two great Circles DBF, DBE, will mutually (*c*) bisect one another in D, B. Therefore each Arc DB, will be a Semicircle. But because a great Circle passing thro' one of the Poles of any Circle in a Sphere, also (*d*) passes thro' the other Pole, and there is a Semicircle of a great Circle interposed between the two Poles; it is manifest, that D being one of the Poles, of the Circle AB, the Point B will be the other Pole: which is absurd. For B is in the Circumference of the Circle. Wherefore the great Circle DB passes thro' E. Q. E. D.

(*a*) 20. 1. *of thes.* (*b*) 4. *of this.* (*c*) 11. *of this.* (*d*) Cor. 10. 1. *of this.*

THEO. VI. PROP. VI.

If a great Circle in a Sphere touches another Circle describ'd in it's Superficies, the said great Circle may also touch another Circle equal and parallel to it.

Fig. 40. LET the great Circle AB, in a Sphere, touch the Circle AC in A. I say the Circle AB may also touch another

another Circle, equal and parallel to AC. For let D, be the Pole of the Circle AC: (*a*) And thro' D, A, describe the great Circle DA: Which, because it passes thro' D, the Pole of the Circle AC, and the Point of Contact A, (*b*) will also pass thro' the Poles of the Circle AB. And assuming E, the other Pole of the Circle AC, draw the right Line DE, (*c*) which will pass thro' the Center of the Sphere. And therefore will be a Diameter of the Sphere. Now from the Pole E, and with the distance EB, describe the Circles BF. I say the great Circle AB, likewise touches the Circle BF in B, and the Circle BF, is equal and parallel to the Circle AC. For because the right Line DE, (*d*) passing thro' the Poles of the Circles AC, BF, is perpendicular to those Circles. (*e*) The Circles AC, BE, will be parallel. (*f*) Again, because great Circles in a Sphere mutually bisect each other, ACB, will be a Semicircle; and so equal to the Semicircle DCE. Therefore the common Arc BD, being taken away, there will remain the equal Arc's DA, EB; (*g*) and therefore right Lines DA, EB, drawn from the Poles D,E, to the Circumferences of the Circles AC, BF, will be equal. (*h*) Wherefore the Circles AC, BF, are equal. Finally, because the Circles AB, BF, cut the great Circle AEB, in which are their Poles, in the Point B, (*i*) they will mutually touch one another in the said Point B. Wherefore the great Circle AB, touching the Circle AC, in a Sphere, also touches the Circle BF, equal and parallel to AC. Q. E. D.

(*a*) 20. 1. *of this.*
(*b*) 5. *of this.*
(*c*) 10. 1. *of this.*
(*d*) 10. 1. *of this.*
(*e*) 14. 11.
(*f*) 11. 1. *of this.*
(*g*) 29. 3.
(*h*) Schol. 2. 1. 1. *of this.*
(*i*) 3. *of this.*

COROLLARY.

From hence it is manifest, that the Points of Contact, A,B, are diametrically opposite. For it has been proved that ACB, is a Semicircle, and accordingly a right Line drawn from A to B, is a Diameter of the Sphere, or of the great Circle ACB.

THEO. VII. PROP. VII.

If there are in a Sphere two equal and parallel Circles: a great Circle, touching one of them, will likewise touch the other.

IN the last Figure let there be two equal and parallel Circles, AC, BF, and let the great Circle AB, touch the Circle AC. I say AB, also touches BF. For if AB, does not touch BF, (*a*) let it touch some other Circle equal and parallel to AC. Therefore since BF, also is equal to AC, and parallel, there will be three Circles in a Sphere, *viz.* AC, BF, and that other which AB, touches equal between themselves, and parallel. Which is absurd. (*b*) For there can be but two Circles, equal, and parallel, in a Sphere. Q. E. D.

(*a*) 6. *of this.*

(*b*) Schol. 2. *of this.*

SCHOLIUM.

The following Theorem is demonstrated in the other Version.

Parallel Circles in a Sphere, which some great Circle touches, are equal betwen themselves.

Still in the last Figure, let there be two parallel Circles AC, BF, which the great Circle AB, touches in A, B. I say the Circles AC, BF, are equal to each other. For because the Circles AC, BF are supposed parallel, (c) they will have the same Poles, which let be D, E; (d) thro' which and the Poles of the Circle AB, let there be describ'd the great Circle AFB, (e) which will pass thro' the Points of Contact A,B. But because great Circles of a Sphere mutually bisect each other, ADB will be a Semicircle, and therefore equal to the Semicircle DBE. Wherefore taking away the common Arc DB, there will remain the Arc's DA, EB, equal; (f) and accordingly right Lines DB, EB, drawn from

(*c*) 1. *of this.*
(*d*) 20. 1. *of this.*
(*e*) 4. *of this.*
(*f*) 29. 3.

from the Poles D,E, to the Circumferences of the Circles AC, BF will be equal. (g) *Wherefore the Circles AC, BF, will be equal.* Q. E. D. (d) Schol. 21. 1. *of this.*

THEO. VIII. PROP. VIII.

If a great Circle in a Sphere be oblique to some other Circle of the Sphere, it may touch two Circles, equal to one another, and parallel to the aforesaid Circle to which it is oblique.

LET the great Circle AB, in a Sphere, be oblique to any Circle, as CD. I say the Circle AB, may touch two equal Circles, and parallel to CD. (*a*) For let E, F, be the Poles of the Circle CD, (*b*) thro' which and the Poles of the Circle AB, let the great Circle EAB, be described, cutting AB, in A,B. Moreover from the Pole E, and with the distance EA, let the Circle AG be described. Then because the Circles AB, AG, cut the great Circle EAB, in which are their Poles, in the Point A, (*c*) they will mutually touch one another in the said Point A. Therefore the great Circle AB, touching the Circle, AG, (*d*) may touch another equal and parallel to it, which let be BH. But because the parallel Circles AG, BH, (*e*) have the same Poles, E,F: And E, F are likewise the Poles of the Circle CD; the three Circles AG, CD, BH, will have the same Poles; (*f*) and therefore they will be parallel between themselves. Wherefore the gteat Circle AB, touches the two Circles AG, BH, equal between themselves, and parallel to CD, which is oblique to the great Circle. Q. E. D.

Fig. 41.
(*a*) 21. 1. *of this.*
(*b*) 20. 1. *of this.*
(*c*) 3. *of this.*
(*d*) 6. *of this.*
(*e*) 1. *of this.*
(*f*) 2. *of this.*

SCHOLIUM.

This Theorem is here added, in the other Version.

If a great Circle in a Sphere, touches some Circle in the same, it will be oblique to those Circles it cuts, which are parallel to the Circle it touches.

In the last Figure, let the great Circle AB, touch the Circle AG, but cut the Circle CD, parallel to AG. I say the Circle AB, is oblique to the Circle CD. For because the great Circle AB, touching the Circle AG, does not pass thro' its Poles (for if it should pass thro'
(a) 15. 1. of this. *its Poles, it (a) would bisect it, and not touch it.) And therefore neither thro' the Poles of the Circle CD; (b)*
(b) 1. of this. *(for the parallel Circles AG, CD, have the same Poles) the great Circle AB, will not cut the Circle CD, at*
(c) 13. 1. of this. *right Angles: (c) Otherwise it passes thro' its Poles. Therefore it is oblique to the Circle CD. Which was proposed.*

THEO. IX. PROP. IX.

If two Circles in a Sphere mutually cut one another, a great Circle drawn thro' their Poles, will bisect the Segments of those Circles.

Fig. 42. (a) 20. 1. of this. LET the two Circles ABCD, EDFB, in a Sphere mutually cut one another, in the Points B, D, and (a) let there be describ'd thro' their Poles the great Circle AFCE, cutting the said Circles, in the Points A, C, E, F. I say the Circle AFCE, bisects the Segments BAD, BCD,
(b) 15. 1. of this. BED, BFD. (b) For because the great Circle AFCE, bisects the Circles ABCD, EDFB, at right Angles, as being drawn thro' their Poles, the common Sections AC, EF, which it makes with them, will be their Diameters crossing one another in G. For the right Lines AC, EF mutually intersect each other, because they are both in the Plan of the Circle AFCE, and the Point F is between the Points A, C; and the Point E, between the same Points. Now draw the right Lines BG, DG; then the three Points B, G, D, will be in the Plans of both the

Circles ABCD, EDFB; and so in their common Section: (c) But their common Section is a right Line. Therefore BGD, will be a right Line. And because the Circle AFCE, has been proved to cut both the Circles ABCD EDFB, at right Angles; both these Circles will reciprocally be at right Angles to the Circle AFCE, (d) and therefore BD, their common Section will be perpendicular to the same. VVherefore the Angles BGA, DGA, BGC, DGC, will be right ones, from *Def.* 3. *lib.* 11. *Euclid.* VVherefore since the Diameter AC, passes thro' the Center of the Circle ABCD, and cuts the right Line BD at right Angles, it (e) will bisect it. Therefore because the Sides AG, GB, are equal to the Sides AG, GD, and contain equal Angles, namely right ones, (f) the Bases AB, AD, subtending the Arc's AB, AD, will be equal, (g) and so likewise the Arc's AB, AD. In the same manner we demonstrate that the Arc's CB, CD, are equal; as also the Arc's EB, ED; and FB, FD. Therefore the Circle AFC, bisects the Segments BAD, BCD, BED, BFD. Q. E. D.

(c) 3. 11.
(d) 29. 11.
(e) 3. 3.
(f) 4. 1.
(g) 28. 3.

SCHOLIUM.

There are here added, in the other Version, these two other Theorems, viz.

I.

If Circles in a Sphere mutually cut one another; some other Circle, bisecting their Segments, will pass thro' their Poles, and be a great Circle.

In the last Figure, let the two Circles ABCD, EDFB, mutually cut one another in the Points B, D, and let another Circle, as AFCE, bisect the Segments BAD, BCD, BED, BFD. I say the Circle AFC, passes thro their Poles, and is a great Circle. For because the Arc's AD, AB, are equal, as also CD, CB; the whole Arc's ADC, ABC, will be equal, and accordingly Semicircles. And in the same manner EDF, EBF, will be Semicircles. Therefore the Circle AFCE, bisects the

the Circles ABCD, EDFB, and ſo the common Sections AC, EF, interſecting each other in G, are their Diameters. Now the right Lines BG, DG, being drawn, becauſe the three Points B, G, D, are in both the Plans of the Circles ABCD, EDFB; and ſo in their common Section,
(a) 3. 11. *(a) which will be a right Line, BGD is a right Line.*
(b) 29. 3. *but becauſe the right Lines DA, DC, are equal to the right Lines BA, BC, becauſe of the equal Arcs, and*
(c) 21. 3. *contain equal Angles, (c) to wit, right ones, as being*
(d) 4. 1. *in Semicircles; (d) the Angles DAC, BAC, will be equal. Which likewiſe may be thus proved Becauſe the Sides DA, AC, are equal to the Sides BA, AC, and the*
(e) 8. 1. *Baſe DC, equal to the Baſe BC, (e) the Angles DAC, BAC, will be equal. Again, becauſe the Sides AD, AG, are equal to the Sides AB, AG, and contain equal Angles, as has been proved; the Angles AGD,*
(f) 4. 1. *AGB, (f) will be equal, and accordingly right ones. Therefore BGD is perpendicular to AC. In the ſame manner, it may be proved, that the ſaid right*
(g) 4. 11. *Line BGD, is perpendicular to EF. (g) Wherefore the ſaid BGD, will be perpendicular to the Plan of the Circle AFCE, drawn thro' the right Lines AC, EF;*
(h) 18. 11. *(h) and accordingly both the Plans of the Circles ABCD, EDFB, drawn thro' the right Line BGD, will be at right Angles to the Plan of the Circle AFCE: Whence reciprocally the Circle AFCE, is at right Angles to the Circles ABCD, EDFB. Therefore the Circle AFCE, will biſect the Circles ABCD, EDFB, at right Angles. (i) Wherefore it will be a great Circle*
(i) Schol. 15. 1. *of this:* *and paſs through their Poles: which was propoſed.*

II.

If two Circles in a Sphere mutually biſect each other, a great Circle biſecting any two of their Segments, not having the Arc interpoſed between thoſe Segments, equal to a Semicircle; will paſs thro' their Poles, and biſect the two other Segments.

Fig. 43. *Let the two Circles ABCD, EBFD, mutually interſect one another in the Points B, D; and let the great Circle*

cle AFCE, cut any two Segments of them, to wit, BAD, BED, in half in the Points A,E, so that the Arc AFCE, intercepted between the said Segments be not a Semicircle. I say the Circle AFCE, passes through the Poles of the Circles ABCD, EBFD, and cuts the other Segments BCD, BFD, in half. For if the Circle ACE, does not pass through their Poles, let there be described, if possible, another great Circle, as AGE, through their Poles, (a) which will bisect their Segments; and so will pass through the Points A, E. (b) Wherefore the great Circles AFCD, AGE, will cut each other in half in A, E: and accordingly AFCE, will be a Semicircle: Which is contrary to the Hypothesis. Therefore the Circle AFCE, passes through the Poles of the Circles ABCD, EBFD. (c) Wherefore all the Segments of them will be bisected. Q. E. D.

(a) 9. *of this.* (b) 11. 1. *of this.* (c) 9. *of this.*

THEO. X. PROP. X.

great Circles in a Sphere are described thro' the Poles of parallel Circles; the Arc's of the parallel Circles, intercepted between the great Circles, are similar; and the Arc's of the great Circles intercepted between the parallel Circles, are equal.

LET there be in a Sphere, the two parallel Circles ABCD, EFGH, (the Pole of which is I; (*a*) (for parallel Circles have the same Poles.) And thro' I, any how describe the great Circles AEIGC, BFIHD. I say the Arc's of the parallels AB, EF, are similar, as also BC, FG; likewise CD, GH; and DA, HE: But the Arc's of the great Circles *viz.* AE, BF, CG, DH being between the parallels, are equal. For let the common Sections of the Circle AIC, and the Parallels be the right Lines AC, EG, (*b*) which will be parallel; and the common Sections of the Circle BID, and the same Parallels, let be the right Lines BD, FH, which likewise will be parallel. Then because the great Circles AIC,

Fig. 44. (*a*) 1. *of this.* (*b*) 16. 11.

AIC, BID, described through the Poles of the Parallels,
(c) 15. 1. *of this.* (c) bisect the said Parallels; AC, BD, will be Diameters of the Circle ABCD, and the Point L, wherein they intersect will be the Center of the same Therefore because the right Lines EK, KF, are parallel to the
(d) 10. 11. right Lines AL, LB, and are in different Plans, (d) the Angles EKF, ALB, at the Centers K, L, will be equal. Wherefore by *Schol. Prop.* 22. *lib*, 3. *Euclid*, they will be similar. And in the same manner, will BC, FG; and CD, GH: as also DA, HE, be similar.
Again, because right Lines drawn from I, to A, B,
(e) 28. 3. C, D, are equal; (e) the Arc's IA, IB, IC, ID, will be equal: And so likewise will IE, IF, IG, IH. Therefore the remaining Arc's AE, BF, CG, DH will be equal. Q. E. D.

THEO. XI. PROP. XI.

If equal Segments of Circles are erected at right Angles, on the Diameters of equal Circles, in the Circumferences of which Segments, are assumed equal Arc's, each of which, reckoning from the Extremity of its Segment, is lesser than half the Circumference of the whole Segment; and if from the Points terminating the aforesaid equal Arc's, are drawn equal right Lines to the Circumferences of the equal Circles, the Arc's of the said Circles, intercepted between those right Lines, and the Extremities of their Diameters, will be equal.

Fig 45. 46. LET the equal Segments AGC, DHF, be at right Angles on the Diameters AC, DE, of the equal Circles ABC, DEF; and assume the equal Arc's AG, DH, so that the Points G, H, may not cut the Segments AGC, DHF, in half. Lastly, let the equal right

Lines

Lines GB, HE, fall on the Circumferences of the equal Circles ABC, DEF. I ſay the Arc's AB, DE, are equal. (*a*) For draw from G, H, the right Lines GI, (*a*) 11. 11. HK, perpendicular to the Plans of the Circles ABC, DEF, (*b*) which will fall in the Po:ts I, K, of the (*b*) 38. 11. common Sections AC, DF. Likewiſe having aſſumed the Centers L, M, of the Circles ABC, DEF, draw the right Lines LB, BI, AG; ME, EK, DH; and firſt, let the Points I, K, fall in the Semidiameters AL, DM. Therefore becauſe the Arc's AGC, DHF, are equal, and alſo the Arc's AG, DH; likewiſe the Arc's CG, FH will be equal; (*c*) and accordingly the Angles (*c*) 27. 3. GAC, HDF ſtanding upon them, are equal. But the Angles AIG, DKH, are alſo equal, as being right ones, from *Def.* 3. *lib.* 11. *Euclid.* Therefore the two Triangles AIG, DKH, have the two Angles GAI, AIG, equal to the two Angles HDK, DKH. (*d*) They have (*d*) 29. 3. likewiſe the Side AG, equal to the Side DH (becauſe of the equality of the Arc's AG, DH.) Therefore (*e*) the (*e*) 26. 1. Side AI, will be equal to the Side DK, and the Side DI, to the Side HK. But becauſe the Angles GIB, HKE are right ones, from *Def.* 3. 11. *Euclid*, (*f*) the Squares (*f*) 47. 1. of GB, HE; which are equal to one another (becauſe of the equality of the right Lines GB, HE) will be equal to the Squares of GI, IB, and of HK, KE. Therefore taking away the equal Squares, of the equal right Lines GI, HK, the Squares of the right Lines IB, KE, will remain equal; and ſo the right Lines IB, KE, are equal. And becauſe the Semidiameters AL, DM, of equal Circles, are equal: and AI, DK, have been proved to be equal, likewiſe IL, KM, will be equal. Wherefore the Sides IL, LB, will be equal to the Sides KM, ME: But the Baſes IB, KE, have been proved equal. (*g*) Therefore the Angles L, M, at the Centers, will be (*g*) 8. 1. equal; (*h*) and accordingly the Arc's AB DE, will be equal. (*h*) 26. 3.

Again, let the Points I, K, fall in the Semidiameters LA, MD, produced towards A, D: Which Fig. 47. 48. may happen, when the Segments AGC, DHF, are greater than a Semicircle; and make the ſame Conſtruction, as before. (*i*) We demonſtrate, as at firſt, that the (*i*) 27. 3. Angles GAC, HDF, are equal; and accordingly (*k*) be- (*k*) 13. 1. cauſe, as well GAC, GAI, as HDF, HDK, are equal to

two right Angles; GAI, HDK, will be equal. And therefore becauſe the Angles at I,K, are equal, *viz.*
(*l*) 29. 3. right ones, (*l*) and the Sides GA, HD, equal, (becauſe
(*m*) 26. 1. of the equal Arc's AG, DH.) The (*m*) right Lines GI, IA, will be equal to HK, KD, as before; and ac-
(*n*) 47. 1. cordingly IL, KM, will be equal. (*n*) Therefore, as
(*o*) 8. 1. at firſt, the right Lines IB, KE, are equal, (*o*) and the
(*p*) 26. 3. Angles L,M, (*p*) and finally the Arc s, AB, DE.

Fig. 49. 50. *Thirdly*, Let the Perpendiculars, drawn from G, H, to the Plans of the Circles ABC, DEF, fall in the Points A, D, which may alſo happen when the Segments AGC, DHF, are greater than a Semicircle. Therefore having drawn the right Lines AB, DE, the Angles GAB, HDE will be right ones, from *Def.* 3. *lib.* 11. *Euclid.* (*q*) Wherefore, as at firſt, the Squares of
(*q*) 47. 1. the right Lines GA, AB, will be equal to the Squares of the right Lines HD, DE: But the Squares of GA, HD, are equal. (*r*) (Becauſe GA, HD, are equal, and
(*r*) 29. 3. the Arc's AG, DH.) Therefore the Squares of AB, DE, will be equal; and accordingly the right Lines AB, DE, are alſo equal. (*s*) VVherefore the Arc's AB, DE, will
(*s*) 28. 3. be equal. Q. E. D.

THEO-

THEO. XII. PROP. XII.

If equal Segments of Circles are set up at right Angles on the Diameters of equal Circles, in the Circumferences of which Segments are assumed equal Arc's, lesser than half the Circumference of the Segments: And if there are taken equal Arc's in the equal Circles, beginning from the Extremities of the Diameters, on the same Side; right Lines drawn from the Points in the Circumferences of the Segments, to the Points in the Circumferences of the Circles, will be equal.

REpeating the Figures of the last Proposition, with the same Constructions, let the Arc's AB, DE, be equal. I say the right Lines GB, HE, are also equal. For because, as in the precedent Propositions has been demonstrated, the right Lines AI, IG, are equal to the right Lines DK, KH; the Lines IL, KM, will be equal. Therefore because IL, LB, are equal to the right Lines KM, ME; and *(a)* contain the Angles at L,M, equal, because of the equality of the Arc's AB, DE; *(b)* the Bases IB, KE, will be equal. Wherefore because the Sides, GI, IB, are equal to the Sides HK, KE and contain the equal Angles GIB, HKE, namely right ones, from *Def.* 3. *lib.* 11. *Euclid.* *(c)* the Bases GB, HE will be equal. VVhich was proposed. This is easily demonstrated when the perpendiculars drawn from G,H, to the Plans of the Circles ABC,DEF,fall in the Points A,D, as in *Fig.* 49. 50. *(d)* For since the right Lines GA, AB, are equal to HD, DE, because of the equal Arc's AG, DH: AB, DE, and contain equal Angles, *viz.* right ones. From *def.* 3. *lib.* 11. *Euclid,* *(e)* the Bases GB, HE will be equal. Q. E. D.

(a) 27. 3.
(b) 4. 1
(c) 4. 1.
(d) 29. 3.
(e) 4. 1.

THEO. XIII. PROP. XIII.

If there are parallel Circles in a Sphere, and great Circles are described which touch one of the Parallels, and cut the others; the Arc's of the Parallels intercepted between those Semicircles of the great Circles, that do not concur, will be similar; and the Arc's of the great Circles intercepted between any two Parallels, will be equal.

Fig. 51. 52. LET there be in a Sphere the parallel Circles AB, CDE, FGH, (a) which will have the same Pole, to wit I. And let the great Circles AFK, BHK, touch the Parallel AB, in the Points A, B, and cut the others in the Points F, C, L, M: H, E, D, G, and themselves in K, N; so that KMN, NFK; KGN, NHK, are Semicircles. (b) For great Circles mutually bisect each other. Also assume the Arc KP, equal to the Arc NB, and KO, equal to the Arc NA, that AMO, OFA, BGP, PHB, may be also Semicircles. Therefore the Semicircles AMO, BHP, do not concur, because they do not mutually cut one another. (These Semicircles are cut off from the Circles AIRO, BITP, as appears in *Fig.* 51. But in *Fig.* 52, the Circles AI, BI, produced thro' R, I, are supposed to pass thro' O, P, that they may cut off the same Semicircles.) In the same manner the Semicircles BGP, AFO, will not concur. Now I say the Arc's of the Parallels AB, LE, MH, intercepted between the Semicircles AMO, BHP, which do not concur, are similar; as also the Arc's AB, CD, FG, intercepted between the non-concurring Semicircles BGP, AFO, are similar: But the Arc's of the great Circles AC, AL, BD, BE, are equal; as also the Arc's CF, LM, DG, EH; whereof the former are interposed between the Parallels AB, CDE, and the latter between the Parallels CDE, FGH: and in the same manner are the Arc's AF, AM, BG, BH, intercepted between the Parallels, AB, FGH, equal.

(a) 1. of this.

(b) 11. 1. of this.

(c)

Fig. 51

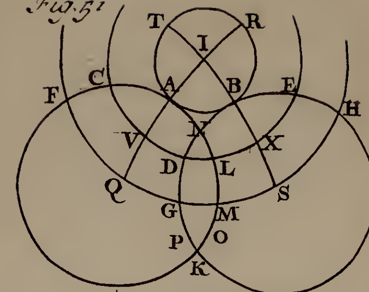


Fig. 52

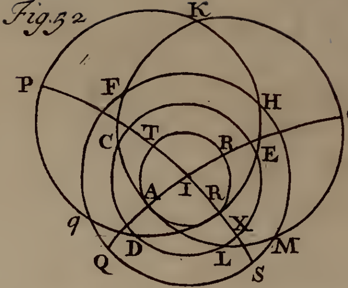


Fig. 54

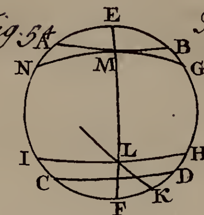


Fig. 55

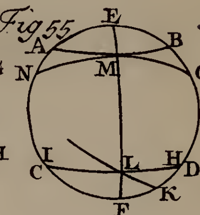


Fig. 56

Fig. 57

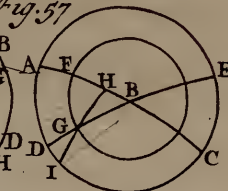


Fig. 58

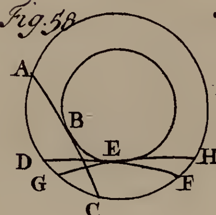


Fig. 59

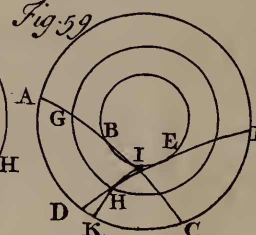


Fig. 60

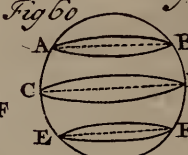


Fig. 61

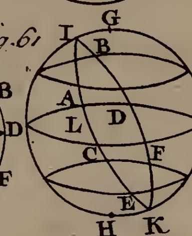


Fig. 62

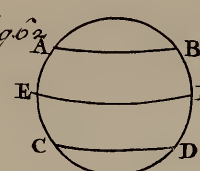


Fig. 63

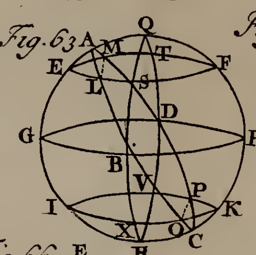


Fig. 64

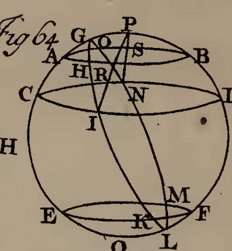


Fig. 65

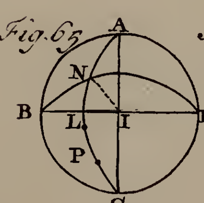


Fig. 66

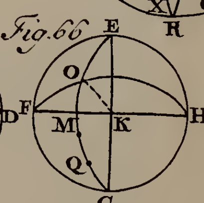


Fig. 67

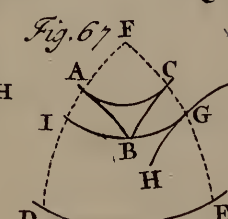
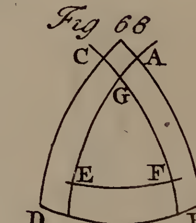


Fig. 68



(*c*) For through the Pole I, and the Points of Contact A, B, describe the great Circles QAIR, SBIT, cutting the Parallel in Q, S, V, X. These great Circles (*d*) will also pass through the Poles of the Circles AFK, BHK; and accordingly (*e*) will bisect the Segments CAL, DBE, CVL, DXE: as also the Segments FAM, GBH, FQM, GSH. (*f*) Besides the said Circles will cut the Parallel AB, CDE, FGH, and the great Circles AFK, BHK at right Angles. Therefore because equal Segments of Circles are at right Angles on the Diameters of the equal Circles AFK, BHK, *viz.* the Semicircles beginning from the Points A, B, and passing through I, until they again cut the Circles AFK, BHK, in the Points O, P, as in *Fig.* 52; (*g*) and the Arc's AI, BI, are equal, (because from the Def. of a Pole, right Lines IA, IB, are such, which are lesser than half the Semicircles: For because they are half the Arc's AIR, BIT, since from the Def. of a Pole, right Lines drawn from I, to the Points A, B, R, T, are equal, and (*h*) therefore also the Arc's are equal: But the Arc's AIR, BIT, are lesser than Semicircles, because the Semicircles tend from A, B, thro' I, to the Circles AFK, BHK; the Arc's AI, BI, will be lesser than half the Semicircles) and also right Lines IC, IE, equal, from the Def. of a Pole, (*i*) the Arc's AC, BE, will be equal. But AC, is equal to AL, and BE to BD, (*k*) because the Arc's CAL, DBE, are bisected, as has been proved. Therefore the four Arc's AC, AL, BE, BD, are equal. We demonstrate in the same manner, that the Arc's AF, AM, BG, BH, are equal; and accordingly also the other Arc's CF, LM, EH, DG, each of which are intercepted between two Parallels. Which was in the second Place proposed to be demonstrated.

(*c*) 20. 1. *of this.*
(*d*) 5. *of this.*
(*e*) 9. *of this.*
(*f*) 15. 1. *of this.*
(*g*) 28. 3.
(*h*) 28. 3.
(*i*) 11. *of this.*
(*k*) 9. *of this.*

Again, because the whole Arc's CAL, DBE, are equal, since their Halves are so, as has been proved; (*l*) Subtenses CL, DE, will be equal, which likewise subtend the Arc's CVL, DXE; (*m*) and accordingly the Arc's of the Parallels CVL, DXE, will be equal. (*n*) Therefore because they are bisected in V, X, as has been said, their Halves will be equal, *viz.* the four Arc's CV, VL, DX, XE. If therefore the common Arc, VD, is added, or taken away, as in *Fig.* 52, to the equal Arc's CV, DX, the Arc's CD, VX, will be equal: (*o*) But the

(*l*) 29. 3.
(*m*) 18. 3.
(*n*) 9. *of this.*
(*o*) 10. *of this.*

the Arc VX, is similar to the Arc AB. Therefore CD, will be similar to the said AB. By the same way of reasoning it may be proved that FG, is similar to the said AB; as also the Arc's EL, HM, are similar to the said AB. Which was first proposed to be proved.

SCHOLIUM.

The non-concurring Semicircles ought to begin from the Points of Contact A,B: Such are AMO, BHP. Wherefore because there are two Semicircles of a great Circle between the Points of Contact of two opposite parallels, the Semicircles of two Circles cutting one another must not be assumed between the Points of Contact of two Parallels, but one must be assumed towards that Point of Section, and the other declining towards the other side; so that the Convexity of one may answer to the Concavity of the other, and contrariwise, as appears in the aforesaid two Semicircles. For if there be taken two Semicircles AMO, DKY, (assuming the Arc KY, equal to DN,) not concurring the Arc's DL, GM, will not be similar. Otherwise two great Circles drawn through the Pole I, and the Points D, L, will pass through the Points G,M: Because, from 10th. of this, they intercept similar Arc's; which cannot be. For DG, LM, are Semicircles: Because by 11th. of the first of this, great Circles bisect one another.

PROB. I. PROP. XIV.

A lesser Circle in a Sphere being given, as also a Point in its Circumference; to describe a great Circle thro' that Point, touching the said lesser Circle.

Fig. 53. LET AB, be a given lesser Circle in a Sphere, whose Pole is C; it is required to draw a great Circle, thro' A, a given Point in its Circumference, which shall touch

touch the Circle AB. (*a*) Describe the great Circle CADEB thro' the Pole C, and the Point A; in which assume the Quadrant AD, and from the Pole D, with the Distance DA, (*b*) describe the Circle AE, which will be a great one, because a Subtense DA, is the Side of a Square inscrib'd in a great Circle. Now I say the great Circle AE, touches the Circle AB, in A. For because the two Circles AB, AE, cut the Circle CAD passing thro' their Poles, in the Point, A, (*c*) they will mutually touch one another in the Point A. Q. E. F.

(*a*) 21. 1. *of this.* (*b*) 17. 1. *of this.* (*c*) 3. *of this.*

PROB. II. PROP. XV.

A lesser Circle in a Sphere being given, and also some Point in its Superficies, which is between the given Circle and another equal and parallel to it; to describe a great Circle thro' that Point, touching the given lesser Circle.

LET AB, be a given lesser Circle in a Sphere, to which CD is equal, and parallel, and let G be the given Point, between the two given Circles AB, CD: It is required to draw thro' G, a great Circle, touching the Circle AB. Let E, F, be the Poles of the Parallels AB, CD, (*a*) (for Parallels have the same Poles) and (*b*) describe thro' E, G, the great Circle EAC, which will pass thro' the other Pole F (*from Coroll. of Schol. Prop.* 10. *lib.* 1. *of this*) in this assume the Quadrant BH; and whither the Point H, falls above D, in D, or below D, (*c*) proceed thus. From the Pole E, with the Distance E,H, or from the Pole F, with the Distance FH, describe the Circle HI, which will be parallel to AB, CD, and be above CD, or the same as CD, or Lastly will be below CD, according as the Point H, is posited above D, in D, or below D.

Fig. 54, 55, 56.

(*a*) 1. *of this.* (*b*) 20. 1. *of this.* (*c*) 2. *of this.*

Again

Again, Assume the Quadrant GK, and the Point K, will be beyond H, because GH, is lesser than a Quadrant. Moreover from the Pole G, with the Distance GK, describe the Circle KL, (*d*) which will be a great one, because a right Line subtending the Quadrant GK is equal to the Side of a Square inscribed in a great Circle. Let KL, cut the Circle HI, in L, (*e*) (for it will necessarily cut it, because the Point K, is below H, and does not come to I. (For because the Parallels AB, CD, are equal (*f*) right Lines EA, FD, will be equal; (*g* and accordingly the Arc's AE, DF, will be equal. Therefore adding the common Arc AF, the Arc's EAF, AFD, will be equal; and consequently since AEF, is a Semicircle between the Poles E, F; AFD will also be a Semicircle. But AI, is a Quadrant; (*h*) because it is equal to the Quadrant PH; wherefore ID will be a Quadrant: and accordingly IG will be greater than a Quadtant. Therefore assuming the Quadrant GK, the Point K, will fall below H, but will not come to I. Whence the Circle HI, is cut by the Circle KL,) and thro' L, F, describe the great Circle FL, which will pass thro' the other Pole E, (from *Corol. Schol. Prop.* 10. *lib.* 1. *of this.*) and let this Circle FLE, cut the Circle AB in M. (*i*) Now the Arc's ML, BH, of the great Circles passing thro' E, F, the Poles of the Parallels, intercepted between the Parallels AB, HI, are equal; and accordingly BH being a Quadrant by Construction LM, will also be a Quadrant. Therefore from the Pole L with the Distance LM, describe the Circle MN, (*k*) which will be a great one, since a right Line subtending the Quadrant LM, is equal to the Side of a Square inscribed in a great Circle. But because the great Circle KL, passes thro' L, the Pole of the great Circle NM, so reciprocally will the great Circle NM (*l*) pass thro' G, the Pole of the Circle KL: and consequently the great Circle NM, will pass thro' the given Point G. Now I say it likewise touches the Circle in M. For because the Circles AB, GN, cut the great Circle GF in the Point M, in which are their Poles, (*m*) they mutually touch one another in M. Therefore there is describ'd thro' G, the great Circle GN, touching the Circle AB in M. Q. E. F.

(*d*) 17. 1. *of this.*
(*e*) 20. 1. *of this.*
(*f*) Schol. 21. *of this.*
(*g*) 28. 3
(*h*) 10. *of this.*
(*i*) 10. *of this.*
(*k*) 17. 1. *of this.*
(*l*) Schol. 15. 1. *of this:*
(*m*) 3. *of this.*

SCHO-

SCHOLIUM.

If the Point G is given exactly in the middle of the Arc BD; GF will be a Quadrant. For then if there are added the Arc's BE, DF, which (n) are equal, to the equal Arc's GB, GD, the Arc's GE, GF, will be equal; and accordingly EGF, being a Semicircle between the Poles E,F; GE, GF, will be Quadrants. Therefore from the Pole G, and with the distance GF, the Circle EF being described, will cut HI, in the Point L, which again will be the Pole of the touching Circle, as before. But if the given Point G, is the same as D, the Pole of the touching Circle will be in the middle of the Arc DCA, because this Arc is a Semicircle. And the Circle described from that Pole, touches AB in A, and CD, in D; since this great Circle, and the Parallels AB, CD, cut the Circumference of the great Circle ACDB, in the Points A, D. (n) 28. 3.

But because, as L, has been proved to be the Pole of the great Circle GN, touching the Circle AB, so also it may be demonstrated, that another Point, in which the great Circle KL, cuts the Circle HI on the other Side is the Pole of some other great Circle, which may pass through G, and touch the Circle AB, in another Point. Whence it is manifest, there may be described two great Circles, through a given Point in a Sphere, between two equal and parallel Circles, which may touch the Circle AB, in two Points.

THEO. XIV. PROP. XVI.

Great Circles in a Sphere, cutting off similar Arc's from parallel Circles, either pass thro' the Poles of those Parallels, or touch some one Parallel.

LET the great Circles in a Sphere ABC, DBE, cut off from the Parallels ADC, FG, the similar Arc's AD, FG. I say the great Circles ABC, DBE, either Fig. 57.

 pass

paſs through the Poles of the Parallels, ADC, FG, or touch ſome one parallel. For either one of them, *viz.* AEC, paſſes through the Poles of the Parallels, and ſo we prove the other paſſes through the ſame, or does not paſs through the Poles of the Parallels, but touches one of them, and ſo we ſhall demonſtrate, the other touches the ſame; or finally, it will not paſs through the Poles of the Parallels, nor touch one of them; which being granted, we conclude that the given great Circles, touch ſome other Parallel, leſſer than the given Parallel. For firſt, let ABC paſs through the Poles of the Parallels. I ſay alſo DBE, paſſes through the ſame Poles, that is, the Point B, in which the great Circles ABC, DBE, cut one another, is the Pole of the Parallels ADC, FG. For if B, is not their Pole, let H be it. Then becauſe the Circle ABC, is ſuppoſed to paſs through their Poles, H will be in the Circumference ABC. (*a*) Through H,G, deſcribe the great Circle HG, cutting ADC, in I. And the Arc's AI, FG, (*b*) will be ſimilar, becauſe they are intercepted between the great Circles AH, HI, deſcribed through the Pole H: But the Arc AD is ſuppoſed ſimilar to the Arc, FG. Therefore the Arc's AI, AD, are ſimilar; and conſequently becauſe they are Arc's of the ſame Circle, they will be equal to one another, the whole to the Part: which is abſurd. Therefore no other Point but B, will be the Pole of the Parallels, if one of the Circles ABC, DBE, *viz.* ABC, be drawn through their Poles. Wherefore if one of the great Circles ABC, DBE, paſſes through the Pole B, of the Parallels, the other will alſo paſs through it.

(*a*) 20. 1. *of this.* (*b*) 10. 1. *of this.*

Fig. 58. 2*dly*, let the two great Circles ABC, DEF, again, cut off from the Parallels ADC, BE, the ſimilar Arc's AD, BE, and neither of them paſs through the Poles of the Parallels, but one of them, *viz.* ABC, touch one of the Parallels, ſuppoſe BE, in B. I ſay alſo the Circle DEF, touches the ſaid BE, in E. For if it does not touch, but cuts it; (*c*) deſcribe through the Point E, in the Parallel BE, the great Circle GEH, touching the Parallel, BE, in E; then Semicircles, one of which is drawn from E, through G, and the other from B, through A, do not concur, as is manifeſt from the Figure of *Prop.* 13. of this Book, and from what is there demonſtrated. (*d*) Therefore the Arc's BE, AG, will be ſimilar;

(*c*) 14. *of this.* (*d*) 13. *of this.*

milar: But the Arc's BE, AD are likewise similar. Wherefore AG, AD, are similar. And accordingly because they are Arc's of the same Circle, they will be equal, the whole, and the Part: which is absurd. Therefore no other great Circle drawn through E, besides DEF, touches the Parallel BE, in E, if ABC touches the same in B. Wherefore if ABC, touches BE, DEF, will also touch BE.

Lastly, let the great Circles ABC, DEF, cut off from the Parallels ADC, GH, the similar Arc's AD, GH; and let neither of them be drawn through the Poles of the Parallels or touch either of them. I say the great Circles ABC, DEF, touch some other Parallel lesser than ADC, GH. For because the great Circle ABC, neither passes through the Poles of the Parallels, nor touches either of them, the great Circle ABC will be oblique to both the Parallels ADC, GH. For if it was at right Angles to it, (*e*) it would pass through their Poles, which is contrary to the Supposition. (*f*) Whence ABC may touch two Circles equal and Parallel to ADC, GH. Therefore let it touch the Parallel BE, which will be lesser than either ADC, or GH; (because ABC, cuts them) and so the other equal and parallel to it, will be lesser than ADC, or GH, and accordingly the Parallels ADC, GH, are posited between those two, that the great Circle AC, touches. I say also DEF, touches the same BE. For if it does not touch it, (*g*) describe through the Point H, which is between the Circle BE, and another equal and parallel to it, the great Circle KH, touching BE, in I; then Semicircles, one of which passes from I, through G, and the other from B, through G, will not concur. (*h*) Therefore the Arc's AK, GH, will be similar: But AD, GH, are similar: Wherefore AK, AD, are similar. And consequently because they are Arc's of the same Circle, they will be equal, the VVhole and the Part. Which is absurd. Therefore no great Circle described through H, besides DEF, touches the Parallel BE, if ABC, touches it in B. VVherefore if ABC, touches the Circle BE; DEF, will also touch BE. Q. E. D.

Fig. 59.

(*e*) 13. 1. *of this.*

(*f*) 8. *of this.*

(*g*) 15. *of this.*

(*h*) 13. *of this.*

SCHOLIUM.

It is manifest that the great Circles ABC, DEF, must so touch the Parallel BE, that their Semicircles proceeding through similar Arc's from the Points of Contact, must not concur. For otherwise the Arc's cut off, will not be similar, as appears from Prop. 13 *of this Book.*

THEO. XV. PROP. XVII.

If, in a Sphere, the Arc's of great Circles intercepted between parallel Circles, and a great Circle parallel to them, be equal, the said parallel Circles will be equal; and those Parallels will be lesser that have the Arc's of great Circles intercepted between them, and a great Circle parallel to them, greater.

Fig. 60. LET the parallel Circles AB, CD, EF, be in a Sphere; and let CD be the parallel great Circle. Now between the Circle CD, and either of the Parallels AB EF, let the equal Arc's AC, CE, of any great Circle ACEFD, be intercepted. I say the Parallels AB, EF, are equal. For let the common Sections of the Parallels, and the Circle ACEFDB, be the right Lines (*a*) 16. 11. AB, CD, EF, (*a*) which will be parallel between themselves. And first, let the great Circle ACEFBD, pass through the Poles of the Parallels. Which being sup- (*b*) 15. 1. *of this.* posed, (*b*) the Circle ACEFDB will bisect the Parallels AB, CD, EF, at right Angles; and so AB, CD, EF, will (*c*) 10. *of this.* be Diameters of the Parallels. (*c*) But because the Arc's AC, BD, are equal, as also the Arc's CE, DF; and AC, is equal to CE; AC, BD, together; will be equal to CE, DF, together: But the Semicircles CABD, (*d*) 11. 1. *of this.* CEFD, are equal: (*d*) Because the great Circles CD, ACEFDB mutually bisect each other. Therefore the remaining

maining Arc's AB, EF, will be equal, *(e)* and accordingly also the right Lines AB, EF, that is, the Diameters of the Circles AB, EF, are equal. Therefore the Circles AB, EF, are likewise equal. *(e)* 29. 3.

Again, let the Arc, AC, be greater than the Arc, CE. I say the Circle AB, is greater than the Circle EF. For the same Construction and Demonstration being supposed, the Arc's AC, BD, as at first, *(f)* will be equal, as also CE, DF. Therefore since AC, is supposed greater than CE, the two Arc's AC, BD, together, are greater than the two Arc's CE, DF, together. Wherefore the remaining Arc AB, taken from the Semicircle CABD, will be lesser than the remaining Arc EF, taken from the Semicircle CE. And accordingly also the right Line AB, that is, the Diameter of the Circle AB, will be lesser than the right Line EF, that is, than the Diameter of the Circle EF, as is by us demonstrated in *Schol. Prop.* 29. *lib.* 3. *Euclid*, when the Arc's AB, EF, are lesser than a Semicircle. Wherefore the Circle AB, will be lesser than the Circle EF. Which was proposed. *(f)* 10. *of this.*

But now, let the great Circle ACEFDB, not pass through the Poles of the Parallels AB, CD, EF; and let again the Arc's AC, CE be equal. I say still the Circles AB, EF, are equal. For let G, H, be the Poles of the Parallels AB, CD, EF, *(g)* and describe through G, H, and the Poles of the great Circle ACEFDB, the great Circle GIHK, *(h)* which will cut the Circle ACEFDB, in two Points, as I, K, at right Angles. Therefore because the great Circle GIHK, passes through the Poles of the great Circles ACEFDB, CD, from Construction, these *(i)* will reciprocally pass through the Poles of that. Wherefore the Points C, D, wherein these two Circles intersect each other, will be the Poles of the Circle, GIHK; (for otherwise both the Circles ACEFD, CD, will not pass through the Poles of the Circle GIHK) and accordingly the right Lines CI, CK, (from the *Def* of a Pole) will be equal, and *(k)* so the Arc's CI, CK, will be equal. But the Arc's AC, CE, by the Hypothesis are also equal. Therefore the remaining Arc's AI, EK, will likewise be equal. Again, because the Semicircle IGK, is equal to the Semicircle GKH; *(l)* (for the Circles ACEFDB, and GIHK, Fig. 61. *(g)* 20. 1. *of this.* *(h)* 15. 1. *of this.* *(i)* Schol. 15. 1. *of this.* *(k)* 28. 3. *(l)* 11. 1. *of this.*

GIHK, mutually bisect each other; and accordingly IGK, is a Semicircle; and the Arc GKH, is a Semicircle, because of the Poles G,H, of the Parallels, taking away the common Arc GK, the remaining Arc's GI, HK, will be equal. Wherefore because the equal Segments of Circles IGK, KHI, (*m*) which are Semicircles, are at right Angles on the Diameter of the Circle ICKD, and the Arc's IG, KH, are equal, and not Quadrants (because G,H, are not the Poles of the Circle ICKD:) And also the Arc's IE, KE, are equal, as has been proved; right Lines GA, HE, (*n*) will be equal. (*o*) Therefore the Circles AB, EF, are equal.

(*m*) 11. 1. *of this.*
(*n*) 12. *of this.*
(*o*) Schol. 26. 1. *of this:*

Lastly, If the Arc AC, be greater than CE; I say the Circle AB, is greater than the Circle EF. For having taken the Arc CL, equal to the Arc CE, the Parallel described through L, will (as just now has been proved) be equal to the Parallel EF: (*p*) But the Parallel AB, is lesser than the Parallel described through L, because it is further distant from the parallel great Circle; and consequently from the Center of the Sphere. Therefore the Parallel AB, is also lesser than EF. Q. E. D.

(*p*) 6. 1. *of this.*

THEO. XVI. PROP. XVIII.

The Arc's of great Circles in a Sphere, intercepted between a great Circle, Parallel to two equal and parallel Circles, and those Parallels, are equal: And those Arc's of a great Circle that are intercepted between a greater Parallel, and a great Circle parallel to it, are lesser.

Fig. 62. LET AB, CD, be two equal and parallel Circles in a Sphere, and EF, a great Circle parallel to them: Now let the great Circle ACD, cut all these parallels. I say the Arc's AE, EC, as also BF, FD, are equal. For if they are not, let AE, be greater. (*a*) Therefore the Circle AB, will be lesser than the Circle CD, which is contrary to the Hypothesis. VVhence the Arc's AE, EC, are equal, as also BF, FD.

(*a*) 17. *of this.*

Now

Now if the Circle AB, be greater than the Circle CD; I say the Arc, AE, is lesser than the Arc EC. For if it be not lesser, it will be equal, or greater. If it be equal, the Circles AB, CD, (*b*) will be equal: if greater, the Circle AB, (*c*) will be lesser than the Circle CD, each of which is contrary to the Hypothesis. Therefore the Arc AE, is lesser than the Arc EC. Q. E. D.

(*b*) 17. *of this.*
(*c*) 17. *of this.*

THEO. XVII. PROP. XIX.

If a great Circle in a Sphere, not passing through the Poles of any Number of Parallels, cuts them, it will be in unequal Parts, except the parallel great Circle, and those Segments of the Parallels intercepted in one Hemisphere, (made by the aforesaid great Circle) which are between the Parallel great Circle and the conspicuous Pole, are greater than a Semicircle: But those which are intercepted between the Parallel great Circle, and the occult Pole, are lesser than a Semicircle: Finally, the alternate Segments of the equal and parallel Circles, are equal.

LET the great Circle ABCD, cut the Parallels EF, GH, IK, in L, M; B, D, and O,P, not passing thro' their Poles, which let be Q, R, and let GH be the parallel great Circle, Q, the conspicuous Pole, and R, the occult Pole in the Hemisphere, which is above the great Circle ABCD, and declines towards F. I say the Circle ABCD, does not bisect the Parallels, except the parallel great Circle GH; (*a*) for it bisects this: And the Segment LFM, between the parallel great Circle and the

Fig. 63.

(*a*) 11. 1. *of this.*

the conspicuous Pole Q, is greater than a Semicircle, and OKP, lesser. If lastly, the Parallels EF, IK, are equal, the alternate Segments LFM, OIP, are equal. (*b*) For through the Pole Q, and the Point B, describe the great Circle Q, BRD; which will pass through the other Pole R, (*from Corol. Schol. Prop.* 10. *Lib.* 1. *of this*) as also through the Point D, (*c*) because it divides both the Circles GBHD, ABCD, in half; but these Circles are cut in half in B,D. Whence the Circle QBRD cuts the Parallel EF, above the Circle ABCD; but the Parallel IK, below the same; as in the Points S, T; V,X. (*d*) And because the Circle QBRD, bisects the Parallels EF, IK; SFT, VKX, will be Semicircles; and accordingly the Arc LFM, will be greater than a Semicircle, and the Arc OKP, lesser. Which was proposed.

(*b*) 20. 1. *of this.*

(*c*) 11. 1. *of this.*

(*d*) 15. 1. *of this.*

Now let the parallel Circles EF, IK, be equal. I say the alternate Segments LFM, OIP, are equal; as also the alternate Segments LEM, OKP. (*e*) For describe the great Circle AGCH, through the Poles of the Parallels and the Poles of the Circle ABCD, (*f*) which will bisect the Segments LAM, OCP. Therefore the Arc's AL, AM, are equal; as also CO, CP. And because the great Circle AGCH, passes through the Poles of the great Circles GH, AC, (*g*) these will reciprocally pass through the Poles of that. Therefore the Points B, D, are the Poles of the Circle AGCH; and accordingly right Lines BA, BC, will be equal (from the *Def.* of a Pole,) and (*h*) therefore the Arc's BA, BC, will be equal: But the Arc's BL, BO, (*i*) are likewise equal; because the Parallels EF, IK, are equal. Wherefore the remaining Arc's AL, CO, are equal: But the Arc's AL, CO, are half of the Arc's LAM, OCP, because, it has been proved that right Lines AL, AM; CO, CP, are equal. Therefore the Arc's LAM, OCP, are equal; (*k*) and accordingly the Subtenses LM, OP, will be equal. (*l*) Wherefore from the equal Circles EF, IK, they cut off equal Arc's, the greater one being LFM, equal to OIP, and the lesser one LEM, equal to OKP. Q. E. D.

(*e*) 20. 1. *of this.*

(*f*) 9. *of this.*

(*g*) Schol. 15. 1. *of this.*

(*h*) 28. 3.

(*i*) 18. *of this.*

(*k*) 29. 3.

THEO-

THEO. XVIII. PROP. XX.

If a great Circle in a Sphere, not passing thro' the Poles of any Parallels, cuts them; those intercepted Arc's of the Parallels in one Hemisphere, which are nigher the conspicuous Pole, are greater than those Arc's of the same Parallels, which are similar to the intercepted Segments further from the conspicuous Pole.

LET the great Circle GHIKLMNO, in a Sphere, cut the Parallels AB, CD, EF, in H, O, I, N; K, M, not passing through the Poles; and let P be the conspicuous Pole upon the Hemisphere GBL, and Q, the occult Pole. I say the Arc OBH, is too big to be similar to the Arc NDI, and NDI, too big to be similar to the Arc MFK. (*a*) For describe the two great Circles PI, PN, through the Pole P of the Parallels, and the Points I, N, cutting the Parallel AB, above the Circle GILN, in R, S,: (*b*) Then the Arc RBS, will be similar to the Arc IDN. Therefore because the Arc OBH, is greater than the Arc RBS, it will be too big to be similar to the Arc NDI. In the same manner we demonstrate that the Arc NDI, is too big to be similar to the Arc MFK, to wit, if through the Pole P, and the Points K, M, two other great Circles are described. Q. E. D.

Fig. 64.

(*a*) 20. 1. *of this.*

(*b*) 10. *of this.*

COROLLARY.

From hence it is manifest that the Arc OBH, is a greater Part of its Parallel AB, than the Arc NDI, is of its Parallel *&c.* Because the Arc RBS, is the same Part of its Parallel, as the Arc IDN is of his, as has been proved.

THEO. XIX. PROP. XXI.

If in equal Spheres great Circles, be inclined to great Circles, that, whose Pole is higher above the lower Circle, will be more inclined: But those Circles whose Poles are equally distant from the Plans of the lower Circles, are equally inclined.

Fig. 65. 66. LET the two great Circles BND, FOH, whose Poles are P,Q, be inclined, in the equal Spheres ABCD, EFGH, whose Centers are I,K, to the great Circles ABCD, EFGH; and let in the first Place, the Pole P, be higher above the Plan of the Circle ABCD, than the Pole Q above the Plan of the Circle EFGH. I say the Circle BND, is more inclined to the Circle ABCD, than FOH, to EFGH: *(a)* For describe through the Poles L,P; M,Q, the great Circles ANC, EOG; and let the right Line BD, be the common Section of the Circles ABCD, BND; the right Line AC, of the Circles ABCD, ANC; and the right Line NI, of the Circles BND, ANC: All which right Lines, will pass through I, the Center of the Sphere, *(b)* because great Circles pass through the same Center. In the same Order, let in the other Sphere, the common Section of the Circles EFGH, FOH, be the right Line FH; of the Circles EFGH, EOG, the right Line EG; and of the Circles FOH, EOG, the right Line OK: All which right Lines will likewise pass through K, the Center of the Sphere. Now because the Circle ANC, passing through the Poles of the Circles ABCD, BND, *(c)* cuts them at right Angles; so reciprocally both the Circles ABCD, BND, will be at right Angles to the Circle ANC, *(d)* and consequently the right Line BD, their common Section, will be perpendicular to the same Circle ANC. Wherefore the Angles AID, NID, will be right ones (from *Def.* 3. *lib.* 11. *Euclid.*) And accordingly AIN, will be the Angle of Inclination of the Circle BND, to the Circle ABCD (from *Def.* 6. *lib.* 11. *Euclid.*) in the same manner

(a) 20. 1. *of this.*
(b) 6. 1. *of this.*
(c) 15. 1. *of this.*
(d) 19. 11.

manner EKO, will be the Angle of Inclination of the Circle FOH, to the Circle EFGH. But because P, the Pole of the Circle BND, is higher above the Circle ABCD, than the Pole Q, of the Circle FOH, is above the Circle EFGH; the Arc CP, will be greater than GQ. For since these Arc's are perpendicular to the Circles ABCD, EFGH, they will measure the Altitudes of the Poles P, Q, above their Circles. But the Arc's PN, QO, are equal, as being Quadrants. (*e*) For the Poles P, Q, are distant from the great Circles BND, FOH, a Quadrant. Therefore the Arc CN, will be greater than the Arc GO; and accordingly the remaining Arc AN, of the Semicircle ANC, will be lesser than the remaining Arc EO, of the Semicircle EOG. (*f*) Wherefore the Angle AIN, will be lesser than the Angle EKO; and accordingly the Circle BND, will be more inclined to the Circle ABCD, than the Circle FOH, is to the Circle EFGH, as we have shewn in the Explication of *Def.* 7. *lib.* 11. *Euclid.*

(*e*) Corol. 16. *of this.*

(*f*) Schol. 7. 3.

Now let the Arc's CP, GQ, be equal, that is, let the Poles P, Q, be equally distant from the Plans of the Circles ABCD, EFGH. I say the Circles BND, FOH, are equally inclined to the Circles ABCD, EFGH. For because the Arc's CP, GQ, are equal, if there are added to them the Quadrants PN, QO, the Arc's CN, GO, will be equal; and accordingly the remaining Arc's AN, NO, taken from the Semicircles, will be equal. (*g*) Therefore the Angles AIN, EKO, will be equal, and accordingly (from *Def.* 7. *lib.* 11. *Euclid.*) similar, or the Inclination of the Circles BND, FOH, to the Circles ABCD, EFGH, will be equal. Q. E. D.

(*g*) 27. 3.

SCHOLIUM.

From hence it is manifest, if the Poles of great Circles inclined to others are equaly distant from the Poles of the great Circles to which they are inclined, the Inclinations are equal. But that Circle whose Pole is nigher to the Pole of another to which it is inclined, has a greater Inclination. For if the Arc's LP, MQ, are equal, GP, GQ, will likewise be equal, (h) because CL, GM, are Quadrants; and therefore the Poles P, Q, of the inclin'd Circles, will be equally distant from the Plans

(*h*) Corol. 16. 1. of this.

 of

of the Circles ABCD, EFGH. Wherefore as in this Prop *has been demonstrated, the Inclinations of the Circles BND, FOH, to the Circles ABCD, EFGH, will be equal. But if the Arc LP, be lesser than MQ, the remaining Arc CP, taken from the Quadrant, will be greater than the Arc GQ, taken from the same Quadrant. Wherefore, as has been proved in this* Prop. *the Inclination of the Circle BND to the Circle ABCD, will be greater than of the Circle FOH, to the Circle EFGH.*

We thus demonstrate the Converse of this Theorem and Scholium.

If great Circles in equal Spheres, are equally inclin'd to great Circles, the Diſtances of their Poles from the Plans of the lowermoſt Circles will be equal: But the Pole of that Circle which is more inclined, is higher. Alſo the Diſtances of the Poles of thoſe Circles, that are equally inclin'd, from the Poles of the Circles to which they are inclin'd, will be equal: But the Diſtance of the Pole of that Circle, which is more inclin'd, from the Pole of the Circle to which it is inclin'd, will be leſſer.

For if the Circles BND, FOH, are equally inclin'd to the Circles ABCD, EFGH, the Angles AIN, EKO,
(i) 26. 3. *will be equal (from* Def. 7. Lib. 11. Euclid.) *(i) and accordingly the Arc's AN, EO, will be alſo equal. Therefore adding the Quadrants NP, OQ, the Arc's AP, EQ, will be equal; and conſequently CP, GQ, will be equal. But if the Circle BND, is more inclin'd to the Circle ABCD, than the Circle FOH, it to the Circle EFGH, the Angle AIN, will be leſſer than the Angle EKO, (as we have ſaid in* Def. 7. Lib. 11. Eu-
(k) Schol. 26. 3. clid.) *(k) Whence the Arc AH, will be leſſer than the Arc FO. Therefore adding the Quadrants NP, OQ, the Arc AP, will be leſſer than the Arc EQ; and accordingly CP, will be greater than GQ.*

Again,

Again, If the Circles are equally inclin'd, the Arc's CP, GQ, as before was demonstrated, will be equal. (l) Therefore because CL, GM, are Quadrants; the Arc's LP, MQ, are equal. (l) Corol. 16. 1. *of this.*

If, lastly, the Circle BND, be more inclin'd, the Arc PC, as just now was proved, will be greater than the Arc GQ. Therefore LP, will be lesser than MQ.

Two other Theorems in the other Version are also here added, viz.

I.

Great Circles touching the same parallel, are equally inclin'd to the parallel great Circle: But that great Circle which touches a greater Parallel, is more inclin'd to the parallel great Circle. And Circles equally inclin'd to the parallel great Circle, touch the same Parallel: And that Circle which has a greater Inclination to the parallel great Circle, touches a greater Parallel.

Fig. 67.

Let the great Circles AB, CB, touch the same Parallel AC; and let DE, be the parallel great Circle. I say the Circles AB, CB; are equally inclin'd to the Circle DE. For let F, be the Pole of the Parallels, (a) and through F, and the Points of Contact A, C, describe the great Circles FAD, FCE, (b) which will pass through the Poles of the Circles AB, CB; (c) and therefore will cut them at right Angles. (a) 20. 1. *of this.* (b) 5. *of this.* (c) 15. 1. *of this.*

Wherefore the Arc's AF, CF, measure the Altitude of the Pole F, of the Circle DE, above the Circles AB, CB; (d) and accordingly since the Arc's AF, CF, are equal, because Subtenses FA, FC, are such (from Def. of a Pole) the Circle DE, (e) will be equally inclin'd to the Circles AB, CB, and these will be reciprocally inclin'd to that. (d) 28. 3. (e) 21. 1. *of this.*

Now let the great Circle GH, touch a greater Parallel GI. I say the Inclination of the Circle GH, to the parallel great Circle, DE, is greater than the Inclination of the Circle AB. (f) For having described (f) 20. 1. *of this.*

through

through F, *and the Point of Contact* G, *the great Circle* FGE, *the Arc* FG, *will measure the altitude of the Pole* F, *of the Circle* DE, *above the Circle* GH. *But the Arc* FG, *is greater than the Arc* FA, *because the Circle* GI, *is supposed greater than the Circle* AC, *and accordingly is more remote from the Pole* F. (g) *Therefore the Circle* DE, *will be more inclined to the Circle* GH, *than to the Circle* AB; *and reciprocally* GH, *will be more inclined to* DE, *than* AB.

(g) 11. 1. of this.

Again, Let the great Circles AB, CB, *be equally inclined to the Parallel great Circle* DE. *I say they touch the same Parallel.* (h) *For through* F, *the Pole of the Parallels, and the Poles of the Circles* AB, CB; *describe the great Circles* FAD, FCE, *cutting the Circles* AB, CB, *in* A, C. (i) *Now because they are cut at right Angles; the Arc's* FA, FC, *measure the altitude of the Pole* F, *of the Circle* DE, *above the Circles* AB, CB. (k) *But the Arc's* FA, FC, *are equal, because the Circles* AB, CB, *are equally inclined to the Circle* DE, *and so reciprocally these to those If therefore from the Pole* F, *with the distance* FA, *or* FC, *the Circle* AC, *is described.* (l) *This will touch the Circles* AB, CB, *because the Circle* AC, *and the Circles* AB, CB, *cut the great Circles* FD, FE, *passing through their Poles, in the same Points* A, C.

(h) 20. 1. of this.
(i) 15. 1. of this.
(k) Schol. 21. of this.
(l) 3. of this.

Lastly, *let the great Circle* GH, *be more inclined to the Circle* DE. *I say it touches the greater Parallel,* (m) *for having described through* F, *the Pole of the Parallels, and the Pole of the Circle* GH, *the great Circle* FG, (n) *which will cut the Circle* GH, *at right Angles,* viz. *in the Point* G; *the Arc* FG *will still measure the altitude of the Pole* F, *above the Circle* GH, (o) *But* FG, *is greater than* FA, *because the Circle* GH, *is more inclined than* AB. *Therefore the Circle described from the Pole* F, *with the Interval* FG, *will be greater than the Circle described from the same Pole* F, *with the distance* FA. (p) *Wherefore because* AB, AC, *mutually touch each other in* A, *and* GH, GI, *also in* G, *the thing proposed is manifest.*

(m) 20. 1. of this.
(n) 15. 1. of this.
(g) Schol. 21. of this.
(p) 3. of this.

II.

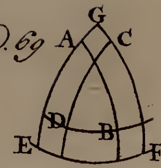


Fig. 69

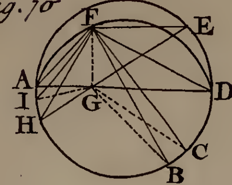
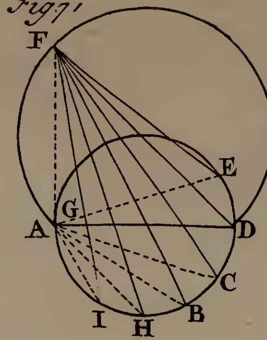


Fig. 70



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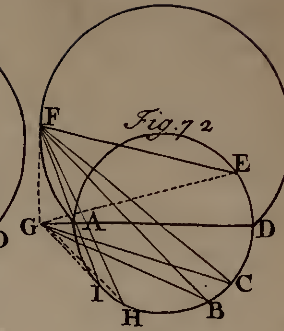


Fig. 72

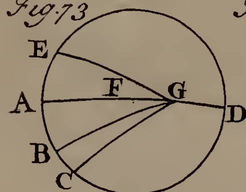


Fig. 73

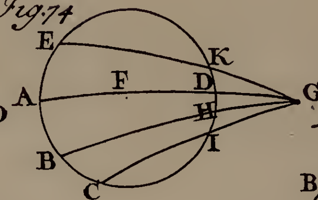


Fig. 74

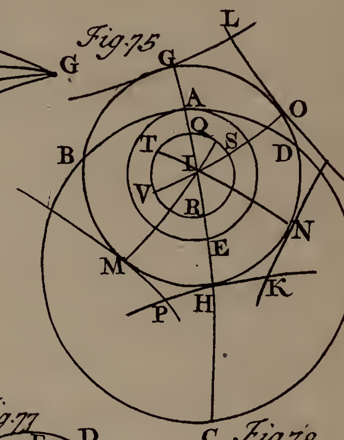


Fig. 75

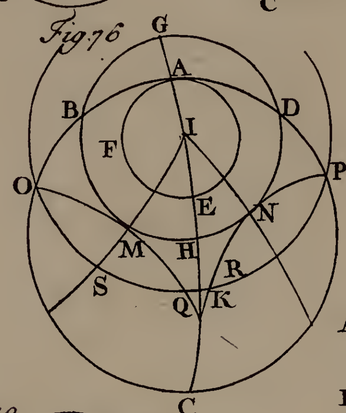


Fig. 76

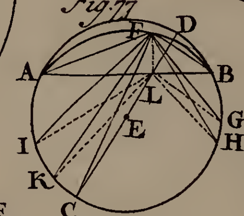


Fig. 77

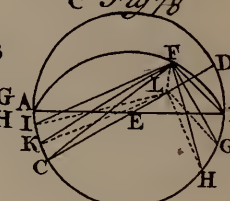


Fig. 78

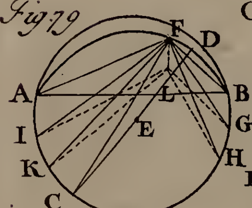


Fig. 79

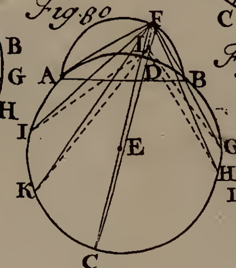


Fig. 80

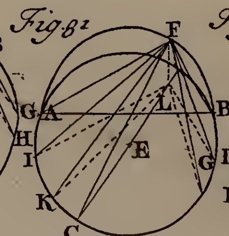


Fig. 81

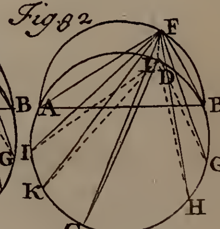


Fig. 82

II.

Great Circles equally inclined to a parallel great Circle, have their Poles in the Circumference of the same Parallel. And great Circles, which have their Poles in the Circumference of the same Parallel, are equally inclined to the Parallel great Circle.

Let the great Circles AB, CD, whose Poles are E, F, be equally inclined to DB, a parallel great Circle. I say their Poles E, F, are in the same Parallel. (a) For having described thro' G, the Pole of the Parallels, and E,F, the Poles of the Circles AB, CD, the great Circles GE, GF, (b) which will be at right Angles to the Circles AB, CD; the Arc's EG, FG, will be the distances of the Poles E,F, from the Pole G: (c) But they are equal, because the Circles AB, CD, equally incline to the Circle DB. Therefore the Circle EF, described from the Pole G, with the distance GE, or GF, (d) is Parallel to the Circle DB; in which parallel EF, are the Poles E,F, of the Circles AB, CD, which was proposed. Fig. 68, 69. (a) 20. 1. of this. (b) 15. 1. of this. (c) Schol. 21. of this. (d) 2. of this.

But now let the great Circles AB, CD, have their Poles E,F, in the Parallel EF. I say they are equally inclined to DB the Parallel great Circle. For, from the Def. *of a Pole, right Lines GE, GF, are equal, (e) and consequently also the Arc's EG, FG. Therefore because the same Arc's, are the distances of the Poles E,F, of the Parallels, from the Pole G, the Circles AB, CD, (f) will be equally inclined to DB, the Parallel great Circle.* (e) 28. 3. (f) Schol. 21. of this.

There here follows in the Greek, *the 22d Proposition, whose Demonstration is very long. Whence because in the other Version the same is shorter and more clearly demonstrated, there are here added three other Theorems, by which the following 22d Proposition may easier be demonstrated. But the first Theorem is the second Part of* Prop. 1. Lib. 3. *of* Theodosius; *tho' as it is here proposed, is more universal. Therefore the first Theorem, which is the third in this Scholium, is this.*

III.

III.

If upon the Diameter of a Circle be constituted at right Angles the Segment of a Circle; and the Circumference of the insistent Segment, be divided into two unequal Parts; and if from the Point of Section, to the Circumference of the first Circle, several Lines be drawn; the right Line subtending the lesser Part of the insistent Segment, will be the least of them all: and that which subtends the greater Part, is the greatest of them all. But of the others, that right Line which is nigher the greatest, will always be greater than that more remote: And that nigher the least, will always be lesser than that more remote. And two equal right Lines which fall from the same Point to the Circumference of the Circle, are equally distant from the greatest right Line.

Fig. 70. 71. 72. *Upon the Diameter AD, of the Circle ABCDE, let the Segment AFD, be erected at right Angles, which is not bisected in F; and let the lesser Part be AF, and the greater DF: and let there fall from F, several right Lines, as FA, FI, FH, FB, FC, FD, FE. I say FA, is the least of them all; FD, the greatest: But FC, is greater than FB, &c. and FI, lesser than FH, &c. Finally, the two right Lines FE, FC, are equal, if they are equally distant from the greatest FD, that is, if the Arc's DE, DC, are equal. (a) For draw from F, to the Plan of the Circle ABCDE, the Perpendicular FG, (b) which will fall in the common Section AD: And the Point G, will be between the Points A,D, as in the first Figure, (which will always happen, when the Segment AFD, is lesser than a Semicircle, and sometimes when it is greater) or be the same as A; or will be without the Circle, in the Diameter DA, produced, as in the two last Figures. Now, in the first Figure, G, will not be the Center of the Circle ABCDE,* be-

(a) 21. 11.
(b) 38. 11.

because GF, does not bisect the Segment AFD: Much less will G be the Center of the Circle ABCDE, in the two last Figures. Draw the right Lines GI, GH, GB, GC, GE; then all the Angles at G, will be right ones (from Def. 3. Lib. 11. Euclid.) *Now (c) because GA is the least of all the right Lines drawn from G, to the Circumference of the Circle ABCDE, in the first and third Figures; and in all the Figures, GD, (d) is the greatest; and GC, greater than GB; and GI, lesser than GH, and Lastly, GC, GE, equal: Whence in the first and third Figures the Squares of the right Lines AG, GF, together, will be lesser, than the Squares of the right Lines IG, GF together: (e) To which because the Squares of the right Lines FA, FI, are equal; the Square of FA, will be lesser than the Square of FI. And so FA, lesser than FI. We prove in the same manner that FA, in the first and third Figures, is lesser than FH, &c. And in the second Figure (f) FA, is also lesser than FI, or FH &c. Because in the Triangles AIF, AHF, (in which the Angle A, is a right one, from* Def. 3. Lib. 11. Euclid, *and so the others acute) the right Line FA, subtends the acute Angle I, or H, but the right Lines FI, FH, &c. the right Angle A. Therefore the right Line FA, is the least of them all. Again, in all the Figures, the two Squares of GD, GF will be greater than the two Squares of GC, GE: (g) To which because the Squares of FD, FC, are equal; the Square of FD will also be greater than the Square of FC, and accordingly the right Line FD, will be greater than FC. So also FD will be greater than FB, &c. Therefore the right Line FD, is the greatest of them all.*

(c) 7. or 8 of 3.
(d) 7. 15. or 8, 3.
(e) 47. 1.
(f) 47. 1.
(g) 47. 1.

Moreover in all the Figures, the two Squares of GC, GF, will be greater than the two Squares of GB, GF: (h) to which because the Squares of FC, FB, are equal; the Square of FC, will be greater than the Square of FB; and so the right Line FC, will be greater than FB. We prove in the same manner, that the right Line FC, which is nigher the greatest FD, is greater than any other more remote, &c. For in all the Figures, the two Squares of the right Lines GI, GF, are lesser than the two Squares of GH, GF: (i) to which because the Squares of FI, FH, are equal; the Square of FI, will

(h) 47. 1.
(i) 47. 1.

also be lesser than the Square of FH; and so FI, will be lesser than FH. We prove thus that the right Line FI, which is nigher the least FA, is lesser than any other more remote, &c. Lastly, *the two Squares of* GC, GF,
(k) 47. 1. *are equal to the two Squares of* GE, GF: (k) *to which because the Squares of* FC, FE, *are equal, the Squares of* FC, FE, *will also be equal; and so the right Lines* FC, FE, *will be equal, Therefore we have demonstrated what was proposed. Again, as from the Demonstration appears. I say that right Line is nigher the greatest* FD, *which falls in a Point nigher to the Point D: And that is nigher to the least* FA, *which falls in a Point nigher the Point* A.

IV.

If a Point be assigned in the Superficies of a Sphere within the Periphery of any Circle, except its Pole, and from that Point to the Circumference of the Circle several Arc's of great Circles lesser than Semicircles are drawn; the greatest is that drawn thro' the Pole of the Circle; and the least that which is adjacent to it: But of the others, that which is nigher to the greatest is always greater than that more remote: And the two Arc's equally remote from the greatest or least, are equal between themselves.

Fig. 73. *Let* ABCDE, *be a Circle in a Sphere, whose Pole is* F, *and assume in the Superficies of the Sphere within the Periphery of the Circle, any Point as* G, *except the Pole* F, *from which let there be drawn any Number of Arc's of great Circles to the Circumference of the Circle* ABCDE, *whereof* GA, *both ways produced, let pass thro' the Pole* F, *and let the Arc* GB *be nigher to* GA, *than* GC; *and Lastly, let* GB, GE, *be equally distant from* GA, *or* GD; *let also all these Arc's be lesser than Semicircles: Which they will be, when they intersect*
(a) 11. 1. of this. *each other in no other Point but* G. (a) (*For because great*

great Circles mutually bisect each other, the Arc's GA, GE, *will be lesser than Semicircles, as not yet intersecting one another. And for the same reason, other Arc's drawn thro'* G, *will be lesser than Semicircles, if they do not mutually intersect each other. But if one of them, as the Arc* GA, *be a Semicircle, all the others will pass thro' the Point A, and will also be Semicircles: But if* GA, *is greater than a Semicircle, all the others will cut it, before they come to the Circumference, and will be greater than a Semicircle from whence nothing can be gathered.) I say the Arc* GA, *is the greatest of all, and* GD, *the least: But* GB, *is greater than the Arc* GC; *Lastly,* GB, GE, *are equal.* (*b*) *For because the Arc* AD, *cuts the Circle* ABC, *in half, and at right Angles; the right Line* AD, *will be the Diameter of the Circle* ABC; *and upon this is erected at right Angles, the Segment* AGD *of a Circle, which is unequally cut in* G, (*for because from the* Def. *of a Pole, the right Lines* FA, FD *are equal,* (*c*) *the Arc's* FA, FD, *will also be equal, and so the Arc* AD, *is bisected in* F. *And therefore in* G *it is not halved) and the greater Part is* GA, *and the lesser* GD. (*d*) *Therefore* GA, *is the greatest of all right Lines drawn from* G *to the Circumference of the Circle* ABC, *and* GD, *the least: But* GB, *is greater than* GC: *And* GB, GE, *are equal. Therefore because the Arc's which they subtend are lesser than Semicircles;* (*e*) *the Arc* GA, *will be the greatest;* GD, *the least:* GB, *greater than* GC; *and lastly,* GB, GE, *are equal.*

(*b*) 15. 11 *of this.*
(*c*) 28. 3.
(*d*) Schol. 21. *of this.*
(*e*) Schol. 28. 3.
(*f*) 28. 3.

V.

If in the Superficies of a Sphere, without the Periphery of any Circle, be assumed a Point except its Pole, and from that to the Circumference of the Circle are drawn any Number of Arc's of great Circles, lesser than a Semicircle, and cutting the Circumference of the Circle; the greatest is that drawn thro' the Pole; and of the others, that which is nigher the greatest, is always greater than that more re-

remote: But the least is that Arc of the greatest, contained between the Point without the Circle, and the Circumference of the Circle; and of others, that which is nigher the least, is always lesser than that more remote: And those two Arc's equally remote from the greatest or least, are equal between themselves.

Fig. 74. *Let ABCDE be a Circle in a Sphere whose Pole is* F, *and assign in the Superficies of the Sphere without the Periphery of the Circle any Point* G, *except the other Pole of the Circle ABCDE: And from* G *let there be drawn any Number of Arc's of great Circles to the Circumference of the Circle ABCDE, cutting it; whereof GDFA, passes thro' the Pole* F; *but the Arc* GHB, *let be nigher to GDFA, than GIC: Lastly, let GHB, GKE, be equally distant from GDFA, or GD; and let them all be lesser than a Semicircle: Which they will be; when they intersect each other in no other Point but in* G, *as has been proved in the precedent Theorem. I say the Arc* G*A, is the greatest of them all;* GB, *greater than GC: But the least is GD; and* GH *is lesser than GI: Finally, the Arc's GB,* GE, *also* GH, GK, *are equal.* (a) 15. 1. of this. *(a) For because the Arc* G*A, bisects the Circle ABCDE at right Angles, AD, will be the Diameter of the Circle ABCDE, and upon this is erected at right Angles, the Segment of a Circle* DG, *which is drawn from D, thro'* G, *till it again cuts the Circle ABCDE, in the Point A. Now this Segment is not bisected in* G *(because* G, *is not the Pole of the Circle ABCDE in which the said Segment is bisected, as has been proved in the precedent Theorem) and the greater Part, is from the Point* G *to A, because the lesser Pole is in that, (otherwise the Arc* GD*A, is drawn thro' both the Poles, and accordingly will be greater than a Semicicle, since the Arc between the two Poles is a Semicircle) but the lesser is DG.* (b) Schol. 21. of this *(b) Therefore* G*A is the greatest of all the right Lines drawn from* G *to the Circumference of the Circle ABCDE: and* GD, *the least; but* GB, *is greater than* GC; GB, GE, *are equal. Also* GH *is lesser than* GI; *and* GH, GK *equal. Wherefore because the Arc's are lesser than a Semicircle, from the Hypothesis* (c) Schol. 28. 3. *(c) the*

Arc

Arc G*A will also be the greatest of them all, and* GD, *the least: But* GB, *is greater than* GC; *and* GH, *lesser than* GI. (*d*) *Finally* GB, GE, *as also* GH, GK, *are* (*d*) *equal.* *Q. E. D.* (d) 28. 3.

It is manifest from the two last Theorems, that the Arc's drawn from G, *ought not to be greater than a Semicircle: Otherwise greater Lines will not cut off greater Arc's, and contrarewise, as is manifest from* Schol. Prop 28. Lib. 3. Euclid.

THEO. XX. PROP. XXII.

If a great Circle in a Sphere touches some Circle, and cuts another parallel to it, posited between the Center of the Sphere, and that Circle which the great Circle touches, and if great Circles are described touching the greater of the two Parallels: All these great Circles will be inclin'd to the first proposed great Circle, and the most erect of them will be that whose Contact is in that Point, in which the greater Segment of the greater Parallel is bisected; But the lowest and most inclin'd, is that whose Contact is in that Point, in which the least Segment is bisected: And of the others, those that are equally distant from either of the Points of them, in which the Segments are bisected, are similarly inclin'd: but that which has a more remote Contact from that Point, in which the greater Segment is bisected, is perpetually more inclin'd to the first mention'd great Circle, than that which has its Contact nigher the same Point. Finally, the Poles of *of the great Circles will be in the same Circle, which also will be lesser than that Circle, which the great Circle first proposed touches, and will be parallel to it.*

Fig. 75. LET the great Circle ABCD, in a Sphere, whose Pole is E, touch the Circle AF, and cut another, as GHBD, parallel to AF, posited between the Center of the Sphere and the Circle AF, so that the Circle GBHD, may be greater than AF; and let E, the Pole of the great Circle ABCD, be between the Circles AF, GBHD. (But because the great Circle ABCD, does not bisect the Circle GBHD, as not passing through its Poles, that is, through the Poles of the Parallels, the Segment BHD, (*a*) will be greater than a Semicircle, and BGD, lesser.) (*b*) Draw through E, the Pole of the Circle ABCD, and I, the Pole of the Parallels, the great Circle GAC, (*c*) which will bisect the Segments BGD, BHD: And let the Points M, N, be equally distant from H; and O further from H, than N; let also the great Circles GL, HK, MP, NK, OL, (*d*) touch the Parallel GBHD, in the Points G, H, M, N, O, all of which will be inclined to the great Circle ABCD, because they do not pass through its Pole E; (for since the Pole E, is supposed between the Parallels AF, GBHD, the Circles touching the Circle GBHD, cannot pass through E, for otherwise they would cut it, because the other Pole, through which they (*e*) must necessarily pass, is without the said Parallels.) I say the Circle HK, is the most erect to the great Circle ABCD; that is, does not incline at all; and the lowest, that is, the most inclin'd, is GL; but MP, NK, are similarly, inclined, and OL, more than NK: Lastly, the Poles of these Circles of contact are in one and the same Parallel, which is lesser than AF. For because E is the Pole of the Circle ABCD, EA (*f*) will be a Quadrant of a great Circle; assume the Arc HQ, equal to it; then the Point Q, will be between the Points A, I, because the Arc HA, is greater than a Quadrant (since EA, has been proved to be one) and HI, lesser than a Quadrant, (*g*) because the Arc drawn from the Pole I, through H, to the parallel great Circle, is a Quadrant. If there-

(*a*) 19. *of this.*
(*b*) 20. 1. *of this.*
(*c*) 9. *of this.*
(*d*) 14. *of this.*
(*e*) Corol. 10. 1. *of this.*
(*f*) Corol. 16. 1. *of this.*
(*g*) Corol. 16. 1. *of this.*

therefore from the Pole I, with the Distance IQ, the (h) 2. *of this.* Circle QTR, be described, (h) it will be parallel to A, F, and lesser than it. Now I say in this Parallel are (i) 20. 1. *of this.* the Poles of all the Circles touching GBHD. (i) For through the Pole I, and the Points of Contact, describe (k) 5. *of this.* the great Circles MIS, NIT, OIV; (k) which will also pass through the Poles of the touching Circles. (l) (l) 28. 3. And because the Arc's HI, MI, NI, OI, GI, are equal, because from the *Def.* of a Pole, the right Lines subtending them are equal, *&c.* For the same Reason, the Arc's IQ, IS, IT, IV, IR, are equal, the whole Arc's HQ, MS, NT, OV, GR, will be equal; and therefore since HQ, is a Quadrant, all those Arc's will be Quadrants. Wherefore because it has been proved, that they pass (m) Cor. 16. 1. *of this.* through the Poles of the contingent Circles, (m) the Points Q, S, T, V, R, will be the Poles of the contingent Circles, all of which will be in the Parallel QTR, which in the last place was proposed to be proved.

Again, because the Arc's of the great Circles drawn from the Pole E, of the great Circle ABCD, to Q, S, T, V, R, the Poles of the contingent Circles, measure the Distances of the Pole E, from the Poles of the contingent Circles; (since these two are equally distant from EQ, because the Arc's QS, QT, are equal. (n) (n) 10. *of this.* For the Arc's of the Parallel VR, between the great Circles HI, MI, NI, are similar to the Arc's MH, NH: And so because these Arc's are equal, those will likewise be equal: Which because they are equal to the Arc's QS, QT; (o) since the common Sections of the (o) 15. 1. *of this.* Parallel VR, and the great Circles HQ, MS, NT, drawn through its Poles, are its Diameters, it is manifest, because the Arc's between these Diameters nigh R, (p) 26. 3. are equal, (p) and also the Arc's QS, QT, opposite to these are equal, that the vertical Angles insisting on the Arc's QS, QT, are equal, and EQ, is (q) the greatest (q) Schol. 21. *of this.* of them all; ER, the least; ES, ET, are equal; and lastly ET, is greater than EN, because all these are lesser than a Semicircle; for EQ, is lesser than the Quadrant EA; and therefore the remaining ones do not cut it about the Point Q: therefore they will be lesser than a Semicircle.) (r) The Circle HK, is not at all (r) Schol. 21. *of this.* inclin'd to the Circle ABCD; and GL, is most in-

clin'd

clin'd; but MK, NK, are equally, or similarly inclin'd; and OL, is inclin'd more than NK. Q. E. D.

THEO. XXI. PROP. XXIII.

The same Things being supposed, if the Arc's of the contingent Circles from the Points of Contact, to the Nodes, are equal; the said great Circles will be similarly inclin'd.

Fig. 76. AGAIN, Let the great Circle ABCD, in a Sphere, whose Pole is E, touch the Circle AF, and cut the Circle GBHD, parallel to it, posited between the Center of the Sphere, and the Circle AF, so that GBHD, may be greater than AF; and let E, the Pole of the great Circle ABCD, be between both the Circles AF, GBHD: Moreover let the great Circles MO, NP, touch the Circle GBHD, in the Points M, N, cutting ABCD, in the Nodes O, P; and let the Arc's MO, NP, be equal. I say the Circles MO, NP, are similarly inclin'd to the great Circle ABCD. (*a*) For draw through E, the Pole of the Circle ABCD, and I, the Pole of the Parallels, the great Circle GAC: Also thro' I, and the Points of Contact, draw the great Circles IM, IN, (*b*) which will also pass thro' the Poles of the contingent Circles, and (*c*) therefore will cut them at right Angles. Wherefore because the equal Segments of Circles, *viz.* the Semicircles which tend from M, and N, thro' I, until they again cut the contingent Circles MO, NP, insist on the Diameters of the Circles MO, NP, (for the common Section of the great Circles IM, MO, will be a Diameter of each Circle, (*d*) because they mutually bisect each other at right Angles, and are not divided in half in I, because I, the Pole of the Parallels, is not the Pole of the contingent Circles; and the Arc's MO, NP, are equal:) (*e*) the right Lines IO, IP, will be equal. If therefore from the Pole I, be described the Parallel OK, with the Distance IO, it will also pass thro' P. And because the great Circle IM, passing thro',

(*a*) 20. 1. *of this.*
(*b*) 5. *of this.*
(*c*) 15. 1. *of this.*
(*d*) 11. 1. *of this.*
(*e*) 12. *of this.*

thro' the Poles of the Circles MO, OQ, cutting one another in O,Q, *(f)* bisects their Segments, the Arc's MO, MQ; SO, SQ, will be equal; and for the same reason will NP, NR, and TP, TR, be also equal; as likewise KO, KP, and CO, CP; because the great Circle IKC passing thro' the Poles of the Circles OKP, OCP, *(g)* bisects their Segments in K,C. Therefore since the Arc's MO, NP, are equal, the Wholes OMQ, PNR, whereof they are the Halves, are equal; *(h)* wherefore the right Lines OQ, PR, will be equal. *(i)* Wherefore also the Arc's OSQ, PTR, will be equal; and accordingly their Halves OS, PT, will be equal. But the Wholes KO, KP, have been proved equal. Therefore the Remainders KS, KT, will be equal; and so since they belong to one and the same Circle, they will be similar between themselves. *(k)* But because the Arc's HM, HN, are similar to the Arc's KS, KT, the Arc's HM, HN, will also be equal. *(l)* Therefore since the Segment BHD, is bisected in H, and the Arc's HM, HN, are equal; *(m)* the Circles MO, NP, will be similarly inclined to the Circle ABCD. Q. E. D.

(f) 9. *of this.*
(g) 9. *of this.*
(h) 29. 3.
(i) 28. 3.
(k) 10. *of this.*
(l) 9. *of this.*
(m) 22. *of this.*

End of the second BOOK.

M THE

THE

The Spherical Elements OF *THEODOSIUS.*

BOOK III.

THEO. I. PROP. I.

If a right Line cuts a Circle into unequal Parts, upon which is erected at right Angles, the Segment of a Circle, which is not greater than a Semicircle; and if the Circumference of the insistent Segment be divided into two unequal Parts: The right Line subtending the lesser of them, is the least of all the right Lines drawn from the Point of Section to the greater Part of the Circumference of the proposed Circle: And of the other right Lines, drawn from the aforesaid Point to the Circumference intercepted be-

between the least right Line, and the Diameter, on which the Perpendicular drawn from the Point falls, that nigher the least, is always lesser than that more remote. But the greatest of them all, is that drawn from the aforenam'd Point to the Extremity of the same Diameter: Also the right Line subtending the greater Arc of the Segment, is the least of those, that fall on the Circumference intercepted between it, and the Diameter, and alway that Line nigher this, is lesser than that more remote. And if the right Line cutting the first named Circle be its Diameter, and all things else, as above; the right Line subtending the lesser Arc of the Segment, is the least of all the right Lines drawn from the Point of Section to the Circumference of the Circle; but that, which subtends the greater Arc of the insistent Segment, is the greatest.

LET the right Line AB, cut the Circle ACBD, whose Center is E, into unequal Parts, whereof let ACB, be the greater: And let the Segment AFB, of a Circle, not greater than a Semicircle, insist at right Angles on AB; the Arc of this Segment let be unequally divided in F; and let BF, be the lesser Part: (*a*) draw from F, to the Plan of the Circle ACBD, the Perpendicular EL, (*b*) which will fall in the common Section; and thro' E,L, draw the Diameter CD; then from F, to the Circumference ACB, of the greater Segment of the Circle ACDB, let there fall the right Lines FB, FG, FH, FC, FA, FI, FK. I say FB is the least of them all, and FG, lesser than FH; but the greatest of them all is FC. Also FA, is the greatest of all those falling from F, on the Portion

Fig. 77.
(*a*) 11. 11.
(*b*) 38. 11.

tion AC; and FI, leſſer than FK. For let there be drawn from L, the right Lines LG, LH, LI, LK; then all the Angles at L, made by the Line FL, (from *Def.*
(c) 7. 3. 3. *Lib.* 11. *Euclid.*) will be right ones. (*c*) Therefore becauſe the right Line LD, is the leaſt of all the right Lines drawn from L, and LB, leſſer than LG, LH, LC, LK, LI, LA, the Squares of FL, LB, together, will be
(d) 47. 1. leſſer than the Squares of FL, LG: (*d*) But the Square of FB, is equal to the Squares of FL, LB; and the Square of FG, equal to the Squares of FL, LG. Therefore the Square of FB, is alſo leſſer than the Square of FG, and conſequently the right Line FB, will be leſſer than FG. In the ſame manner we demonſtrate, that the right Line FB, is leſſer than FH, FC, FK, FI, FA. Wherefore FB is the leaſt of them all.

(e) 7. 3. Again, (*e*) becauſe LG, is leſſer than LH, the Squares of FL, LG, are leſſer than the Squares of FL, LH:
(f) 47. 1. (*f*) But the Square of FG, is equal to the Squares of FL, LG, and the Square of FH, equal to the Squares of FL, LH. Therefore the Square of FG, will be leſſer than the Square of FH; and conſequently FG, will be leſſer than FH.

(g) 7. 3. Further, (*g*) becauſe LC, is the greateſt of all the Lines drawn from L; the Squares of FL, LC, are grea-
(h) 47. 1. ter than the Squares of FL, LK. (*h*) But the Square of FC, is equal to the Squares of FL, LC, and the Square of FK, to the Squares of FL, LK. Therefore the Square of FC, will be greater than the Square of FK; and accordingly the right Line FC, will alſo be greater than the right Line FK. In the ſame manner we prove, that the right Line FC, is greater than FI, and FA. Therefore the right Line FC, is the greateſt.

(i) 12. (*i*) Becauſe LA, is leſſer than LI, LK, LC; the Squares of FL, LA, will be alſo leſſer than the Squares
(k) 47. 1. of FL, LI. (*k*) But the Square of FA, is equal to the Squares of FL, LA, and the Square of FI, to the Squares of FL, LI. Therefore the Square of FA, will be leſſer than the Square of FI; and ſo the right Line FA, will alſo be leſſer than FI. In the ſame manner, the right Line FA, may be proved to be leſſer than FK, FC. Therefore EA is the leaſt of all the right Lines drawn from F, to the Arc AC.

Laſtly,

Lastly, (*l*) because LI, is lesser than LK; the Squares of FL, LI, will be lesser than the Squares of FL, LK; but the Square of FI, is equal to the Squares of FL, LI, and the Square of FK, equal to the Squares of FL, LK. Therefore the Square of FI, will be lesser than the Square of FK, and so the right Line FI, will be lesser than the right Line FK. (*l*) 7. 3.

If the right Line AB, bisects the Circle ABCD, so that it may be its Diameter, we have already demonstrated in *Theorem* 3*d.* of *Schol. Prop.* 21. of the precedent Book, that the right Line FB, is the least, and FA, the greatest. Wherefore it is not necessary to prove the same thing here.

THEO. II. PROP. II.

If a right Line cuts off the Segment of a Circle, which is not lesser than a Semicircle, and upon that right Line there insists another Segment of a Circle, which is not greater than a Semicircle, and inclined to the former Segment; and if the Circumference of the insistent Segment be divided into unequal Parts; a right Line subtending the lesser Part of the Circumference, is the least of all the right Lines drawn from the Point of Division, to that Arc of the first proposed Circle, which is not lesser than a Semicircle: And all the others follow, as in the precedent Proposition.

LET the right Line AB, cut off from the Circle ACBD, whose Center is E, the Segment ACB, not lesser than a Semicircle, but equal, as in the first Figure, or greater, as in the others; and upon the right Line AB, let there be constituted another Segment of a Circle Fig. 78. 79. 80

Circle AFB, not greater than a Semicircle, but either equal, as in the last three Figures, or lesser, as in the first two Figures, and inclin'd to the other Segment ADB, which is not greater than a Semicircle, because ACB, is supposed equal, or greater than a Semicircle. Also divide the Circumference AFB, in unequal parts in F, and let FB be the lesser part. Now from F, let fall the Perpendicular FL, to the Plan of the Circle ACBD, which will fall either in the Segment ADB, or without it, or else in the Circumference ADB. Again, through the Center E, and the Point L, draw CD, and from F, let the right Lines EB, FG, *&c:* fall to the Circumference ACB. I say FB, is the least of them all; and FG, lesser than FH: The greatest of them all is FC: Also FA, is the least of all those Lines, drawn from F, to the Circumference AC; and FI, lesser than FK. For draw from L, the right Lines LB, LG, LH, LA, LI, LK, and all the Angles at L, which the Perpendicular FL, makes, will be right ones (from *Def.* 3. *lib.* 11. *Euclid.*) (*a*) Therefore because the right Line LD, is the least of them all (which will be nothing in that Figure where the Points L, D, coincide) and LB, lesser than LG, LH, LC, LK, LI, LA, and LC, is the greatest of them all, *&c.* We demonstrate, as in *Theo.* precedent, that the right Line FB, is the least, and FG, lesser than FH: Also FC is the greatest, and FA, the least of all the right Lines falling from F, on the Circumference AC; and FI, is lesser than FK. Q. E. D.

(*a*) 7. 8. 15. of 3.

SCHOLIUM.

Fig. 81. 82. *These two Figures are added, that all the Cases of the Cadence of the Perpendicular may be seen. For in* Fig. 78. *the insistent Segment AFB, is a Semicircle, and FL, falls within the Segment ADB: But in* Figure 82, *FL, falls on the Circumference ADB, the insistent Segment AFB, being a Semicircle; like as in* Fig. 80. *the same insistent Segment AFB, being a Semicircle, the Perpendicular FL, falls without the Segment ADB.*

THEO-

Fig. 83

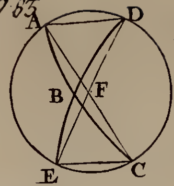


Fig. 84

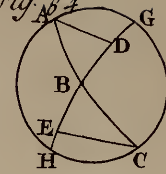


Fig. 85

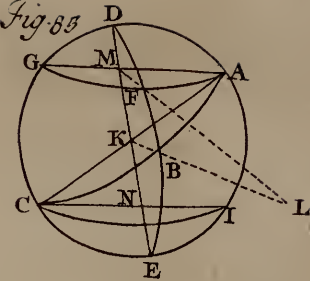


Fig. 86

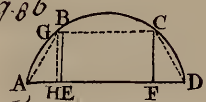


Fig. 87

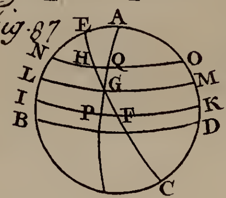


Fig. 88

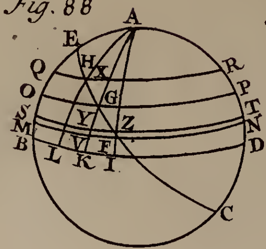


Fig. 89

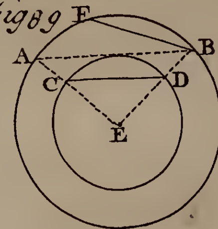


Fig. 90

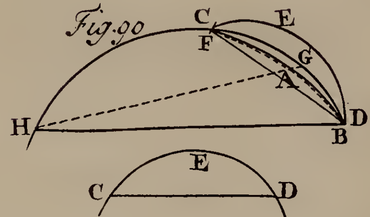


Fig. 91

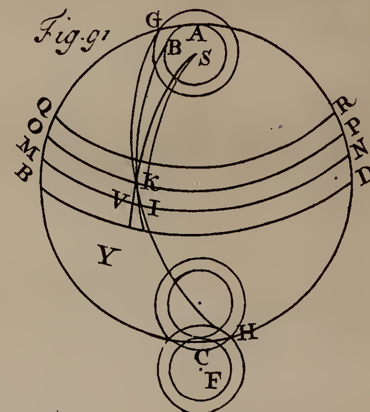


Fig. 92

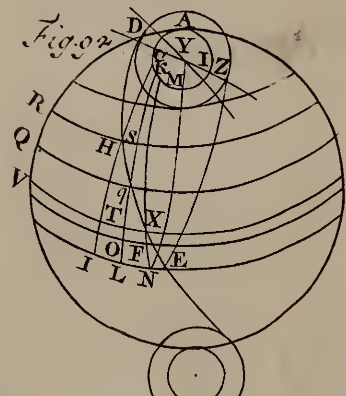
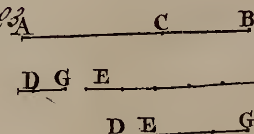


Fig. 93



THEO. III. PROP. III.

If two great Circles in a Sphere mutually cut one another, and if in each of them equal Arc's are assumed on each Side the Point in which they cut one another; Right Lines connecting the extreme Points of these assumed Arc's, on the same Side, are equal between themselves.

LET the two great Circles, in a Sphere, ABC, DBE, Fig. 83. mutually cut each other in B, and in each of them 84. on both sides B, assume two equal Arc's as BA, BC, and BD, BE, and draw the right Lines AD, CE. I say the right Lines AD, CE, are equal. For from the Pole B, and with the Distance BA, describe a Circle, which will also pass through C, because of the equality of the Arc's BA, BC. Therefore the same Circle either passes likewise thro' D, and consequently through E, or not: First, let it pass through D, E, as in the first Figure, and let the right Lines AC, DE, be the common Sections of the great Circles, and of the Circle ADCE. And because the great Circles ABC, DBE, passing thro' B, the Pole of the Circle ADCE, *(a)* bisect it, AC, *(a)* 15. 1. DE, will be Diameters of the Circle ADCE, and F, the *of this.* Center; and accordingly the right Lines FA, FD, are equal to FC, FE. *(b)* And because the vertical Angles *(b)* 15.1 at F, are also equal; *(c)* the right Lines AD, CE, will be *(c)* 4. 1 equal.

Now let the Circle described from the Pole B, with the Distance BA, not pass through D, but beyond it, and so excurs beyond the Point E. But if the Circle AGCH, should pass on this Side the Point D, (which would happen, if the Arc BD, was greater than BA) the Circle must be described from D, with a greater Distance than the Arc BD, that it may excur beyond the Point A. Produce the Arc's BD, BE, to G, H. *(d)* *(d)* 28. 3. Therefore because the Arc's BG, BH, are equal, since from

from the *Def.* of a Pole, Subtenses BG, BH, are equal: And BD, BE, from the Hypothesis, are equal; the remaining Arc's DG, EH, will be equal. And because right Lines AG, CH, are equal, as has been proved in
(*e*) 28. 3. the first Part of this *Prop.* (*e*) the Arc's AG, GH, will be equal. Therefore because the great Circle GBH,
(*f*) 15. 1. *of this.* drawn through the Pole B, (*f*) bisects the Circle AGCH, at right Angles, the Segment GH, insists at right Angles, on the Diameter of the Circle AGCH. Wherefore since the Arc's DG, EH, are equal, and lesser than half the Arc GDH; and the Arc's GA, HC,
(*g*) 12. 2. *of this.* have been proved to be equal, (*g*) the right Lines DA, EC, will be equal. Q. E. D.

THEO. IV. PROP. IV.

If two great Circles in a Sphere mutually cut each other, and in either of them are assumed equal Arc's on each Side the Point in which they intersect; and if through the Points terminating the equal Arc's, there are drawn two parallel Plans, one of which meets the common Section of the Circles, produced without the Sphere towards tht aforesaid Point; and if one of those equal Arc's be greater than either of the Arc's intercepted between the aforesaid Point in the assumed great Circles and the parallel Plans: The Arc, which is between that Point, and the parallel Plan not meetting the common Section of the great Circles, is greater than that Arc of the same Circle, which is between the same Point, and the parallel Plan meeting the common Section of the great Circles.

LET

LET ABC, BDE, be two great Circles in a Sphere, mutually cutting one another in B, assume the equal Arc's BA, BC, and through A, C, let there be drawn parallel Plans, (*a*) making the Circumferences of Circles AFG, CHI, in the Superficies of the Sphere, cutting the Circumference DBE, in the Points F,H; and let the Arc BA, or BC, be greater than either of the Arc's BF, BH, intercepted between the Point B, and the two parallel Plans. Again, from the Pole B, and with the distance BA, or BC, describe the Circle ADCE, which will pass beyond the Points F,H, because the Arc's BF, BH, are supposed lesser than the Arc's BA, BC. Moreover produce the Arc's BH, BF, to the Points D,E, towards the Circumference of the Circle ADCE; and let the common Sections of the Circle ADCE, and the Circles AFG, CHI, be the right Lines AG, CI; and the common Sections of the great Circles, and the Circle ADCE, let be the right Lines AC, DE; which will be Diameters of it, and so the Center will be K, (*b*) because great Circles passing thro' the Pole B, bisect ADCE: Likewise let the right Line DE, cut the right Lines AG, CI, in M,N. Also let the common Section of the great Circles, be the right Line KB, which produced on the Side of B, let meet the Plan AFG, produced without the Sphere, in the Point L. This being supposed, the other Plan CHI, will not meet the right Line KB, on the Side of B, because it does not meet the Plan AFG, parallel to it. I say the Arc BH, is greater than the Arc BF. For let the right Lines FM, HN, be the common Sections of the Circle DBE, and the Circles AFG, CHI, then because the Plan AFG, produced meets the right Line KB, produced in L, the Point L, will be in each of the Plans DBE, AFG; and consequently in their common Section, *viz*, in the right Line MF. Therefore MF, produced, will meet with KB, produced in L. But because the Plan DBE, cuts the parallel Plans AFG, CHI, (*c*) the Sections MF, NH, will be parallel. Again, because the Plan ADCE, cuts the parallel Plans, the Sections AG, CI, (*d*) will be Parallel. (*e*) Therefore the alternate Angles KAM, KCN, (*f*) are equal. But the vertical Angles AKM, CKN, are likewise equal, and the Sides KA, KC, because they are Semidiameters of the Circle ADCE. (*g*) Therefore will

Fig. 85.
(*a*) 1. 1. *of this.*
(*b*) 15. 1. *of this.*
(*c*) 16. 11.
(*d*) 16. 11.
(*e*) 29. 1.
(*f*) 15. 1.
(*g*) 26. 1.

will the Sides KM, KN, be also equal: But the Semidiameters KD, KF, are equal. Therefore the remaining right Lines DM, EN, will be also equal. Again, because the right Line BK, drawn from the Pole B of the Circle ADCE, to K the Center of the same, *(h)* is at right Angles to the Plan of the Circle, the Angle MKL, in the Triangle KLM, will be a right one, from *Def.* 3. *lib.* 11. *Euclid.* *(i)* Therefore the Angle KML, will be an acute one. *(k)* Wherefore because the two Angles FMN, HNM, are equal to two right ones; the Angle HNM, will be obtuse. Therefore, as we shall prove in the following Lemma, the Arc EH, will be lesser than the Arc DF; and so *(l)* because the Arc's BD, BF, are equal, since their Subtenses BD, BE, from the *Def.* of a Pole, are such, the Arc BH, will be greater than the Arc BF. Q. E. D.

(*h*) Schol. 8. 1. of *this.* (*i*) 17. 1. (*k*) 29. 1. (*l*) 28. 3.

LEMMA.

That the Arc EH, is lesser than the Arc DF, we easily prove, this proposed Theorem *being first demonstrated.*

If, too any right Line subtending an Arc of a Circle, two Perpendiculars are drawn from the Arc, cutting off from the Ends of the Arc two equal Arc's, the same two right Lines will cut off from the aforesaid Subtense to equal right Lines. And if two Perpendiculars are drawn to the Subtense of an Arc from the said Arc, cutting off equal right Lines, the said Perpendiculars will cut off two equal Arc's.

Fig. 86. *Let the right Line AD subtend the Arc ABCD, of a Circle, to which from the Arc are let fall the Perpendiculars BE, CF, cutting off the two equal Arc's AB, DC. I say they cut off equal right Lines AE, DF. For having drawn the right Line BC, (m) AD, BC, will be Parallel, because of the Equality of the Arc's AB, DC: (n) also BE, CF are parallel. Therefore BEFC,* is

(*m*) Schol. 27. 3. (*n*) 28. 3.

is a Parallelogram, (o) *and so the right Lines BE, CF,* (o) 34. 1.
are equal. (p) *And because the right Lines AB, DC,* (p) 29. 3.
subtending equal Arc's AB, DC, are equal; the Squares of AB, DC, will be equal. (q) *Wherefore since the* (q) 47. 1.
first is equal to the Squares of AE, BE, and the latter to the Squares of DF, CF; if there are taken away, the equal Squares of the right Lines BE, CF, the Squares of the right Lines AE, DF, will be equal: and consequently the Lines themselves will be equal. Which was the first thing proposed to be demonstrated.

But now let the Perpendiculars BE, CF, cut off the equal right Lines AE, DF. I say they cut off equal Arc's AB, DC. For *if they be not equal, let if possible the Arc AB, be greater than CD, from which cut off AG equal to DC, and from G, to AD, draw the Perpendicular GH: Therefore (as has been proved just now) the right Line AH, will be equal to DF; and consequently to the Line AE: The part to the whole. Which is absurd. Wherefore the Arc AB, is not greater than DC: And for the same reason it will neither be lesser. Therefore it is equal. Which was proposed. From hence it is manifest that the Arc HE, in the Figure of the Proposition, is lesser than the Arc DF. For since the Angle FMK, is acute, and HMK, obtuse, if from M,N, Perpendiculars are drawn to DE, they will fall on the Arc's DF, BH, and will cut off equal Arc's, as we have demonstrated. Wherefore the Arc HE, is lesser than the Arc DF.*

THEO. V. PROP. V.

If the Pole of parallel Circles in a Sphere be in the Circumference of any great Circle, and two other great Circles cut this great Circle at right Angles, one of which is one of the Parallels, and the other is oblique to the Parallels; and if in this oblique Circle equal Arc's are successively taken on the

 same

same Side of the parallel great Circle, and thro' those Points terminating the equal Arc's are described parallel Circles: The Arc's of the first proposed great Circle intercepted between the Parallels will be unequal, and that which is nigher the parallel great Circle, will always be greater than that more remote.

Fig. 87. LET A, the Pole of parallel Circles in a Sphere, be in the Circumference of the great Circle ABCD, and let the two great Circles BD, EC, cut it at right Angles, whereof BD, is the greatest of the Parallels, and EC, oblique to the Parallels: And thro' the Points F, G, H, which cut off from the oblique Circle the equal Arc's F G, GH, describe from the Pole A, the Parallels IK, LM, NO. I say the the Arc IL, is greater than the Arc LN. (*a*) For thro' the Pole A, and the Point G, describe the great Circle AP, cutting the parallels in P, Q. Therefore because there is taken on the Superficies of the Sphere, within the Periphery of the Circle IK, the Point G, besides the Pole A, and from G two Arc's GP, GF, of great Circles fall in the Circumference of the Circle IK; (*b*) the Arc GP, will be the least of them all, and therefore lesser than GF: Because the Arc's GP, GF, are lesser than a Semicircle, since they do not intersect before they divide the parallel IK. For since GP, is a part of a Quadrant tending from A, thro' G, to the Parallel great Circle BD, it cannot cut the Arc GF, beyond the Circle IK, unless GP be either a Semicircle, or greater, and then it will cut GF, in F, or on this Side F. Again, because the Point G is taken in the Superficies of the Sphere without the Periphery of the Circle, and is not in the Circles Pole; (*c*) the Arc GQ, will be the least of all those following from G, that is, lesser than GH: Because the Arc's GQ, GH, are lesser than a Semicircle, since they do not interfect each other before they meet the Parallel NO, which is demonstrated, as before of the Arc's GP, GF. Therefore each Arc

(*a*) 20. 1. *of this.*

(*b*) Schol. 21. 2. *of this.*

(*c*) Schol. 21. 2. *of this.*

Arc FG, GH, is greater than GP, or GQ. And because a right Line drawn thro' G, and the Center of the Sphere, that is, the common Section of the great Circles AP, EC, cuts the Plan of the parallel IK, within the Sphere; (for this right Line will not come to the Center of the Sphere, that is, to the Center of the great Circle ABD, without first cutting the Plan of the Circle IK; since the Parallel IK, is posited between the Parallel great Circle and the Point G.) The said right Line will cut the Plan of the parallel NO, without the Sphere, if they be produced on the Side of G: Since the Point G is posited between the greatest of the parallels and the parallel NO. Therefore because the two great Circles AP, EC, mutually intersect in G, and in the Circle EC, on both Sides the Point G, two equal Arc's GF, GH, are assumed, and thro' F,H, parallel Plans of Circles are drawn, as IK, NO, whereof NO, meets the common Section of the great Circles, AP, EC, without the Sphere, as has been proved, and each of the Arc's GF, GH, is greater than GP, or GQ: *(d)* the Arc GP, will be greater than the Arc GQ: *(e)* but the Arc GP, is equal to the Arc IL, and the Arc GQ, to the Arc LN. Therefore the Arc IL, will be greater than the Arc LN. Q. E. D.

(d) 4. *of this.*
(e) 10. 2. *of this.*

THEO. VI. PROP. VI.

If the Pole of parallel Circles in a Sphere, be in the Circumference of some great Circle, and two other great Circles cut this great Circle at right Angles, one of which is one of the Parallels, and the other oblique to it; and if there are assumed equal Arc's successively on the same Side of the Parallel great Circle, and through the Points terminating the equal Arc's, and the aforesaid Pole, great Circles are described: These will intercept

tercept unequal Arc's of the parallel great Circle, whereof that which is nigher the great Circle first proposed, will always be greater than that more remote.

Fig. 88. LET A the Pole of parallel Circles in a Sphere, be in the Circumference of the great Circle ABCD, and let the two great Circles BD, EC, cut it at right Angles, whereof BD, is the parallel great Circle, and EC, oblique to the Parallels; in which assume the equal Arc's FG, GH; and through the Points F, G, H, *(a)* and the Pole A, let there be described the great Circles AI, AK, AL, cutting BD, in I, K, L. I say the Arc KL, is greater than the Arc IK. For describe thro' the Points F, G, H, the Parallels MN, OP, QR, cutting AK, in V, G, X. *(b)* Therefore the Arc MO, is greater than the Arc OQ; and consequently, *(c)* because the Arc VG, is equal to the Arc MO, and the Arc GX, equal to OQ; the Arc VG, will be greater than GX. Assume the Arc GY, equal to GX, and through Y, describe the Parallel ST, cutting the Circle AI, in Z. Therefore because the Arc's GY, GX, are equal, as also GF, GH, *(d)* right Lines HX, YF, will be equal. And because the great Circle AI, passing through the Pole A, *(e)* bisects the Circle ST, at right Angles, the common Section, *viz.* the Line drawn from Z to the other Section, will be a Diameter of the Circle ST, upon which insists at right Angles to the Circle AI, a Semicircle, to wit, the Semicircle beginning from the Point Z, and going through S to the other Section (that is, the Segment of a Circle, not greater than a Semicircle:) and that right Line cuts off from the Circle AI, a Segment greater than a Semicircle, *viz.* which is drawn from the Point Z, through I, to the other Section with the Circle ST, and YZ, an Arc of the insistent Semicirle, is lesser than a Quadrant, (because the Arc IK, *(f)* which is similar to it, is also lesser than a Quadrant; which thus may be demonstrated. Since the great Circles BD, EC, are right to the great Circle ABCD, this likewise will be right to those, and consequently will pass thro' their Poles. Wherefore it *(g)* will bisect their Segments,

(a) 20. 1. *of this.*
(b) 5. *of this.*
(c) 10. 2. *of this.*
(d) 3. *of this.*
(e) 10. 2. *of this.*
(f) 10. 2. *of this.*
(g) 11. 1. *of this.*

ments, (*h*) which are Semicircles, that is, it will divide them into Quadrants. Therefore the Arc of the Circle BD, posited between B, and that Point wherein the Circles BD, EC, cut one another, is a Quadrant, and so IK, is lesser than a Quadrant. For the Circle AK, falls between the Points B, I, since it passes through the other Pole of the Circle ABCD.) And so the remaining Arc of the insistent Semicircle intercepted between Y, and the other Section with the Circle AI, is greater than a Quadrant; a right Line YZ, (*i*) is the least of all the right Lines falling from Y, on the Circumference ZI; and so is lesser than YF, that is, than XF, which we have proved to be equal to the right Line YF. Wherefore because the Circle QR, is lesser than the Circle ST, a greater right Line HX, cuts off a greater Arc from its Circle, than a lesser right Line YZ, from his, as we shall by and by demonstrate. Therefore the Arc HX, is too big to be Similar to the Arc YZ. (*k*) But the Arc KL, is similar to the Arc HX, and IK to YZ. And therefore KL is too big to be similar to IK; and accordingly since they are in the same Circle, the Arc KL, will be greater than the Arc IK. Q. E. D.

(*h*) 9. 2. *of this.*

(*i*) 1. *of this.*

(*k*) 10. 2. *of this.*

LEMMA.

That the right Line HX, cuts off a greater Arc from its Circle than the right Line YZ, from his, will be manifest, the following Theorem being first demonstrated.

Equal right Lines cut off, from unequal Circles, unequal Arc's; and the Arc of the lesser Circle is too big to be similar to the Arc of the greater Circle.

Let AB, CD, be unequal Circles described about the same Center E, and let there be drawn from E, two right Lines, as EA, EB, cutting the Circle CD in the Points C, D, the Arc's AB, CD, (a) will be similar, since the same Angle E at the Center insists on them. And because the right Lines EA, EB, are proportionably cut in the Points C, D, because EA, EB, are equal, as be also EC, ED; (b) the right Lines AB, CD, will be parallel. (c) And so

Fig. 89.

(*a*) Schol. 33. 6.

(*b*) 2. 6.

(*c*) Corol. 4. 6.

so the Triangles EAB, ECD, are similar, having the Angles EAB, ECD, equal, as also EBA, EDC, and
(d) 4. 6. *the Angle E common. (d) Wherefore as EA is to AB;*
(e) 14. 5. *so is EC, to CD: but EA is greater than EC. (e)*
(f) 1. 4. *Therefore AB, will be greater than CD. (f) Where-*
(g) Schol. 28. 3. *fore apply BF, in the Circle AB, equal to CD; (g) then the Arc AB, will be greater than the Arc FB. Wherefore since the Arc CD, is similar to the Arc AB, the Arc CD will be too big to be similar to FB. Q. E. D.*

From hence it is manifest, that much more a greater Line cuts off from a lesser Circle, an Arc too big to be similar to that, which a lesser Line cuts off from a greater Circle. For because the right Line CD, equal to FB, cuts off the Arc CD, too big to be similar to the Arc FB; much more a greater Line than CD, will cut off an Arc too big to be similar to the Arc FB; since (h) that cuts off a greater Arc, than CD. Where-
(h) Schol. 28. 3. *fore in this 6th* Proposition, *the right Line HX, being greater than YZ, cuts off from the lesser Circle QR, the Arc HX, too big to be similar to to the Arc YZ, which the right Line YZ cuts off from the greater Circle ST.*

But this Demonstration is only to be understood of Arc's lesser than a Semicircle: as are BF, CD. For otherwise the Angle in the Center E, will not be common; which notwithstanding is required in the Demonstration. But yet, if a lesser Arc of a lesser Circle be too big to be similar to a lesser Arc of a greater Circle, much more too big will a greater Arc of a lesser Circle be, to be similar to a lesser Arc of a greater Circle. And if it should happen that the right Line CD, cuts off a Semicircle from the lesser Circle, that is, is its Diameter, it is manifest that the Semicircle of the lesser Circle is too big to be similar to a lesser Arc of the greater Circle; neither then will there be any need of a Demonstration.

Fig. 89. *This Lemma being demonstrated, we likewise easily prove, that equal right Lines cut off from unequal Circles, unequal Arc's, that is, Arc's of unequal lengths, so that the Arc of the lesser Circle is longer than the Arc of the greater Circle, and also too big to be similar to it. For let the right* Lines *CD, BF, be equal, and CD cut off an Arc of a lesser Circle CED, and FB, an Arc of a greater Circle, as FGB. I say the Arc*

CED, is larger than the Arc FGB. For the right Line CD, agreeing to FB, the Arc CED, necessarily falls without the Arc FGB; and so the Arc CED, will be longer than the Arc FGB, since that contains this quite within itself, and they are both Arc's concave on the same Side, and have the same exreme Points, as in the Suppositions before Lib. 1. de Sphera & Cylindro Archimedis. *Neither will the Arc CED, coincide with the Arc FGB, or fall within it. For if it is said to coincide with it, the whole Circumference of the Circle CED, will also coincide with the whole Circumference of the Circle FGB, and so the Circles will be equal. Which is absurd, since they are supposed unequal; and if the Arc CED, is said to fall within the Arc FGB, as the Arc CAD. Because, as has been just now proved, the Arc CED, that is, CAD, is too big to be similar to the Arc FGB, assume the Arc HFB, similar to the Arc CAD, and consequently greater than the Arc FGB: And having taken the Point A, any where in the Arc CAD, draw the right Lines AF, AB, and produce FA, till it cuts the Arc FGB, in B: Draw the right Lines GH, GB. Therefore because the Arc's CAD, HFB, are similar, the Angles CAD, HGB, being in those Segments are equal. (i) But because the Angle CAD, is greater* (i) 16. 1. *than the Angle CGB, the external than the internal; and the Angle CGB, also greater than the Angle HBG, the Whole than the Part; the Angle CAD, will be much greater, than the Angle HGB. Which is absurd. For it has been proved equal to it. Therefore the Arc CED, does not fall within the Arc FGB: It neither coincides with it, as has been demonstrated. Wherefore it falls without FGB, and so the Arc CED, will be longer than the Arc FGB, as was said.*

From hence it is also extremely manifest, that much more a greater Line cuts off from a lesser Circle an Arc longer than that, which a lesser Line cuts off from a greater Circle.

THEO. VII. PROP. VII.

If a great Circle in a Sphere touches two parallel Circles, and another great Circle is oblique to them, and touches parallel Circles greater than them, and if their Contact be in the great Circle first proposed, and there are assumed equal Arc's in the oblique Circle, on the same Side the parallel great Circle; if lastly, thro' the Points terminating the equal Arc's parallel Circles be drawn: These will intercept unequal Arc's in the first proposed great Circle, whereof that which is nigher to the parallel great Circle, will be greater than that more remote.

Fig. 91. (*a*) 6. 2. *of this.* LET the great Circle ABCD, in a Sphere, touch the Circle AE, in the Point A; (*a*) and so another, as CF, equal to it: And let another great Circle, as GH, oblique to the aforesaid Parallels, touch two other parallel Circles greater than those, which ABCD, touches, and let the Points of Contact in the great Circle ABCD, be G,H; also let BD, be the parallel great Circle: *Lastly*, assume the equal Arc' IK, KL, in the oblique Circle GH, and thro' the Points I, K, L, let (*b*) 20. 1. *of this.* there be described the parallel Circles MN, OP, QR. I say the Arc MO, is greater, than the Arc OQ. (*b*) For thro' K,S, the Poles of the Parallels, describe the great Circle SK, cutting the Parallels in the Points T,V: (*c*) 15. 2. *of this.* (*c*) Also thro' K describe the great Circle KE, touching the Parallel AE, in E, and cutting the other parallels in X,Y; yet so, that these Points X,Y may be between the (*d*) Schol. 15. 2. *of this.* Points L, T, and V, I. Which may be done. (*d*) Because thro' K, two Circles can be described cutting the Circle AE, whereof one falls between the Arc's KG, KS, and the other without them; (for if they both should touch the Circle AE on the same Side, they would mutually

mutually cut one another near to the Points of Contact, since they would meet one another. VVhich is absurd; because they intersect in a Point opposite to K, between the other Pole and the parallel great Circle. Therefore one of them may touch the Circle AE, on the right Side of KS, which bisects the Circle AE, and the other on the left Side, falling between KG; and KS: as is KE. For if it should fall without KG, it could not touch the parallel AE; because it does not first meet KG, unless in a Point opposite to K, where they mutually bisect one another.) If the first is assumed, the Points X, Y, may fall between the Points L, T, and V, I, since it may cut both KG, KS, in K. Therefore because in the Superficies of the Sphere within the Periphery of the Circle MN, the Point K, is assigned, without its Pole S, and from K, three Arc's KV, KY, KI, fall on its Circumference; (*e*) KV, will be the least of them all, and KY lesser than KI. Again, because in the Superficies of the Sphere without the Periphery of the Circle QR, the Point K, is assigned, without its Pole, and from K, to the Circumference, the three Arc's KT, KX, KL, fall; (*f*) KT, will be the least of them all, and KX, lesser than KL. Therefore each Arc KI, KL, is greater than KY, or KX. And because a right Line drawn thro' K, and the Center of the Sphere, that is, the common Section of the great Circles GH, EY, cuts the Plan of the parallel QR, without the Sphere, if they be produced on the Side of K, (as in the Demonstration of *Prop.* 5. of this Book, has been said,) (*g*) the Arc KY, will be greater than the Arc KX. (*h*) But the Arc MO, is equal to the Arc KY, and the Arc OQ, to the Arc KX; for they are non-concuring Semicircles, whereof one, is drawn from A thro' B, and the other from E, thro' K, (as is manifest from *Prop.* 13. *lib.* 2. of this.) VVherefore the Arc MO, will be greater than the Arc OQ. Q. E. D.

(*e*) Schol. 21. 2. *of this.*

(*f*) Schol. 21. 2. *of this.*

(*g*) 4. *of this.*

(*h*) 13. 2. *of this.*

THEO. VIII. PROP. VIII.

If a great Circle in a Sphere touches two parallel Circles, and another great Circle oblique to them, touches parallel Circles greater than the first mention'd Parallels, and their Contact be in the great Circle first proposed; and if there be taken in the oblique Circle equal Arc's, on the same Side of the parallel great Circle, and through the Points terminating the equal Arc's are described great Circles, which likewise touch the same Circle that the great Circle first proposed touches, and intercept similar Arc's of the Parallels, and have those Semicircles, which tend from the Points of Contact, to the Points terminating the equal Arc's of the oblique Circle, through which they are described, so, that they do not meet that Semicircle of the first proposed great Circle, in which the Contact of the oblique Circle between the apparent Pole, and the parallel great Circle is: They intercept unequal Arc's on the Circumference of the parallel great Circle, whereof that nigher the great Circle first proposed, is always greater than that more remote.

Fig. 92. (*a*) 6. 2. *of this.* LET the great Circle AB, in a Sphere, touch the Circle AC in A, (*a*) and so another equal and parallel to it and let another great Circle DE, oblique to the two Parallels, touch two greater Parallels; and let the Contact, as the Point D, be in the Circle AB; let BE, be

be the parallel great Circle ; and in the oblique Circle DE, aſſume the equal Arc's FG, GH ; and through the Points F,G,H, deſcribe the great Circles CI, KL, MN, touching the Parallel AC, in C, K, M, and cutting BE, the parallel great Circle, in I, L, N, ſo that they may intercept ſimilar Arc's of the Parallels, and their Semicircles, beginning from the Points C, K, M, and paſſing through F, G, H, may not meet the Semicircle AB, beginning from A, and paſſing through B. I ſay the Arc IL, is greater than the Arc LN. For deſcribe through F, G, H, the Parallels PF, QG, RH, cutting the Circle KL, in O, S. *(b)* Therefore the Arc PQ, will be greater than the Arc QR ; *(c)* to which, ſince the Arc's GO, GS, are equal, the Arc GO, will be greater than GS. Make GT, equal to GS, and through T, deſcribe the Parallel VT, cutting the Circle MN, in X. And becauſe the common Section of the Circles MN, VX, that is, the right Line drawn from the Section X, to the other Section, cuts off a Segmenr, beginning from X, and paſſing through V, to the other Section, leſſer than a Semicircle ; *(d)* (for the great Circle MN, cutting the Parallel VX, and not paſſing through its Poles, cuts off a Segment greater than a Semicircle, *viz.* which is between the parallel great Circle, and the conſpicuous Pole, as is the Segment beginning from X, and paſſing through A, to the other Section with the Circle MN,) and cuts off from the great Circle MN, a Segment greater than a Semicircle, *viz.* which beginning from X, paſſes through N, to the other Section ; and the Segment XV, is inclin'd to the Segment XM. (For if through N, Y, the Pole of the Parallels, the great Circle YN, is deſcribed, *(e)* it will be at right Angles to BE. Therefore MN, which is poſited between theſe two, is inclined to the ſaid BE, towards the Parts R; and ſo reciprocally BE, and its Parallel VX, will be inclin'd towards the ſame Parts.) Alſo the Segment beginning from X, and paſſing through V, to the other Section, is cut unequally in T, and the leſſer part is TX, as preſently ſhall be proved. *(f)* Therefore a right Line TX, is leſſer than a right Line TF : But the right Line TF, *(g)* is equal to HS ; and ſo, as in *Lemma Prop.* 6. of this Book is demonſtrated, the Arc HS is too big to be ſimilar

(b) 7. *of this.*
(c) 13. 2. *of this.*
(d) 19. 2. *of this.*
(e) 15. 1. *of this.*
(f) 2. *of this.*
(g) 3. *of this.*

(*h*) 13. 2. *of this.* lar to the Arc TX. (*h*) Therefore since the Arc IL, is similar to the Arc HS, and the Arc LN, to the Arc TX, the Arc IL will also be too big to be similar to the Arc LN; whence because they are in the same Circle, IL will be greater than LN. Q. E. D.

LEMMA. I.

We thus demonstrate that the Arc TX, is lesser than half the Segment beginning from X, and passing thro' V, to the other Section. Thro' E, describe the great Circle EZ, touching the Parallel AC, in Z, which is on the right Side of the great Circle NY: (i) Schol. 15. 2. of this. *(i) Since from E, two Circles touching AC, may be described, one on the left Side of the Circle NY, and the other on the right: And EZ will be a Quadrant. For the great Circle ZY, described thro' Y, the Pole of the Circle AC, and Z, the Point of Contact,* (k) 5. 2. of this. *(k) also passes thro' the Pole of the Tangent Circle EZ.* (l) 9. 2. of this. *(l) Wherefore the Circle YZ, will bisect the Segments BE, EZ.* (m) 11. 1. of this. *(m) Therefore since these great Circles bisect each other, the Segments beginning from the Point E, and passing thro' Z, to the other Section, will be cut in Z, into two Quadrants; and so EZ will be a Quadrant. In the same manner ED will be a Quadrant, if thro' the Pole Y, and the Point of Contact D, the great Circle YD is described.* (n) Cor. 16. 1. of this. *(n) But the Arc of the great Circle between E, and the Pole Y, is also a Quadrant. Therefore the great Circle described from E, as a Pole, with the Distance EZ, will pass thro' the Points Y, D. By the same way of Reasoning NM, may be proved to be a Quadrant; and so the great Circle described from the Pole N, with the Distance NM, passes thro' Y, the Pole of the Parallels, and consequently cuts the Arc BD, beyond the Point D, and the Arc NB, beyond the Arc DB, and so the Arc XV, beyond the same Arc DB: since the great Circles ZYD, MY, mutually cut one another in the Pole Y; and the Point M is beyond the Circle DYZ. But because the great Circle MY, drawn thro' Y, the Pole of the Parallel AC, and M, the Point of Contact,* (o) 5. 2. of this. *(o) will also pass thro' the Pole of the Tangent Circle MN; it will pass thro' the Poles of the Circles XV, MN, cutting each other in X;* (p) 9. 2. of this. *(p) wherefore*

fore it will bisect their Segments. Therefore since it cuts the Segment, beginning from X, and passing thro' V, to another Point in which the Circles XV, NM, intersect each other, beyond the Point V; the Arc XV, is lesser than half the Segment beginning from X, and passing thro' V, to the other Section; whence TX, will be much lesser than half of the same Segment. Which was to be demonstrated. That the Point of Contact M, is without the great Circle DYZ, we thus demonstrate. Because the Arc of the greatest of the Parallels EB, between E, and the Circle YD, (q) is a Quadrant, as also the Arc of the same between N, and the Circle YM; and the Point N, is beyond E, towards B; the Circle YM, will be also without YD; and accordingly M, is without YD.

(q) Cor. 16. 1. *of this.*

LEMMA II.

Two unequal Magnitudes being given: to find another mean one, which may be commensurable to any other given Magnitude.

Let AB, AC, be two unequal Magnitudes given, and also DG any other; it is required to find another mean one, that is, one greater than AC, but lesser than AB, and commensurable to DG. In the first Place, let DG, be lesser than BC, the excess between the Magnitudes AB, AC; and E, a Multiple of DG, the nighest greater than AC. Which being granted, E will be lesser than AB. For if it was equal, if there should be taken from E, a Magnitude equal to DG (which is supposed lesser than BC) there would still remain a Multiple of DG, greater than AC. Therefore E would not be a Multiple of DG, the nighest greater than AC. Which is absurd. Wherefore E, is not equal to AB, and so much more will it not be greater. Therefore it is lesser than AB, and consequently since it is also greater than AC, and commensurable to DG, because it is a Multiple of it, what was proposed is manifest. Fig. 93.

But now let the given Magnitude DG, be not lesser than BC. Therefore DG, being bisected, and again its

(a) 1. 10. *its half bisected, and so continually, (a) till there remains the part DF, lesser than BC; let E be a Multiple of DF, the nighest greater than AC; than E, will*
(b) 12. 10. *be commensurable to DF; (b) and so to DG. Because both E, and DG, are commensurable to DF. Again, in the same manner, as before was demonstrated, E, will be lesser than AB. Therefore since it is also greater than AC, and commensurable to DG, the thing proposed is manifest.*

THEO. IX. PROP. IX.

If the Pole of Parallel Circles in a Sphere, be in the Circumference of a great Circle, which two other great Circles cut at right Angles, one of which Circles is one of the Parallels, and the other oblique to the Parallels: And if there are assumed equal Arc's, in the Periphery of the oblique Circle, which are not continuus, but yet are on the same Side of the parallel great Circle, and if thro' the Pole and each of those Points terminating the equal Arc's, great Circles be described; they cut off from the Periphery of the parallel great Circle, unequal Arc's, whereof that which is nigher to the great Circle first proposed, is always greater than that more remote.

Fig. 94. 95. 96. LET A, the Pole of parallel Circles in a Sphere, be in the Circumference of the great Circle AB, which two great Circles BC, DC, cut at right Angles, whereof BC, is the parallel great Circle, and DC, oblique to the Parallels; in which assume the non-continuus equal
(a) 20. 1. of this. Arc's EF, GH; (a) And thro' the Points E, F, G, H, and

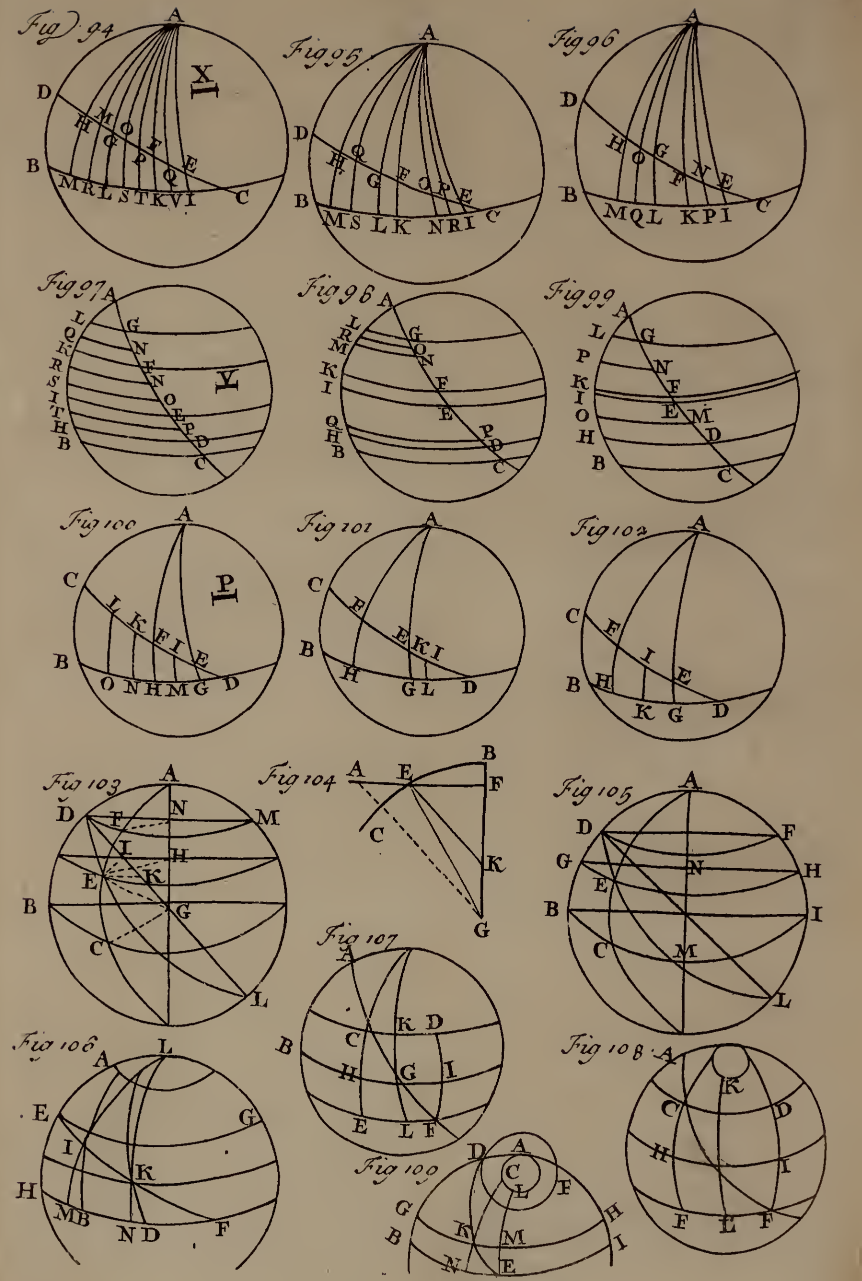


Fig. 94
Fig 95
Fig 96
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Fig 98
Fig 99
Fig 100
Fig 101
Fig 102
Fig 103
Fig 104
Fig 105
Fig 106
Fig 107
Fig 108
Fig 109

and the Pole A, let there be described the great Circles AEI, AFK, AGL, AHM. I say the Arc ML, is greater than the Arc KI. For the intermediate Arc FG, is either commensurable to the equal Arc's EF, GH, or incommensurable. If in the first place it be commensurable, *(b)* having found the greatest common Measure X, divide the three Arc's EF, FG, GH, into Parts equal to X; *(c)* and through the Points of Division, and the Pole A, draw great Circles. Therefore because the Arc's EQ, QF, FP, *&c.* are equal, *(d)* the Arc MR, will be greater than the Arc RL, and RL, greater than LS, *&c.* Wherefore since MR, is greater than KV, and RL, greater than VI, the Whole ML, will be greater than the Whole KI; which was proposed. *(b)* 4. 10. *(c)* 20. 1. *of this.* *(d)* 6. *of this.*

Now let the intermediate Arc FG, be incommensurable to the Arc's EF, GH: I say the Arc ML, is greater than the Arc KI. For if it be not greater, it is either lesser or equal: First, if possible, let ML be lesser than KI: and in KI, assume KN, equal to ML; *(e)* and thro' N, A, describe the great Circle AON, cutting the Circle CD, in O. Moreover, (by *Lemma* 2. aforegoing) find the Arc FP, greater than FO, but lesser than FE, and commensurable to FG; let also GQ, be equal to FP (which is lesser than EA, and so also lesser than GH, equal to EF,) and thro' P, Q, A, *(f)* describe the great Circles APR, AQS. Therefore because the noncontinuus Arc's PF, GQ, are equal, and the intermediate Arc FG, is commensurable to each of them; the Arc SL, will (as has been demonstrated in the first Part) be greater than the Arc KR. Therefore also it will be much greater than KN; and consequently ML, will be much greater than KN: But KN, is equal to ML. Which is absurd. Therefore ML, is not lesser than KI. *(e)* 20. 1. *of this.* *(f)* 20. 1. *of this.*

Lastly, let, if possible, the Arc ML, be equal to KI. And having bisected the Arc's EF, GH, in the Points N, O, *(g)* describe thro' N, O, A, the great Circles ANP, AOQ. *(h)* Therefore the Arc MO, will be greater than the Arc QL, and KP, greater than PI. Wherefore QL, will be lesser than half of MLK; and KP, greater than half KI. Therefore since ML, KI, are supposed equal, QL, will be lesser than KP. VVhich is absurd. For because the Arc's FN, GO, equal to half of the equal Arc's EF, GH, QL, cannot be lesser *(g)* 20. 1. *of this.* *(h)* 6. *of this.*

than KP, (as in the second Part of this Demonstration has been shewn.) Therefore the Arc ML, is not equal to KI: Nor lesser, as has been proved, therefore it is greater. Q. E. D.

SCHOLIUM.

Like as Theodosius *in this Proposition, has demonstrated the same of non-continuus Arc's, as of continuus ones in* Prop. 6, *so in the other Version, there are demonstrated in three other* Theorems *of non-continuus Arc's what* Theodosius *has proved of continuus ones, in* Prop. 5. 7. *and* 8. *The first of the* Theorems *is this.*

I.

If the Pole of parallel Circles in a Sphere be in the Circumference of a great Circle, which two other great Circles cut at right Angles, one of which Circles is one of the Parallels, and the other oblique to the Parallels; and if in this oblique Circle be assumed equal Arc's, which are not continuus, but yet are on the same Side of the parallel great Circle; and there are described parallel Circles thro' each of the Points terminating the equal Arc's. The Arc's of the great Circle first proposed, intercepted between the Parallels, will be unequal, and that which is nigher to the parallel great Circle, will always be greater, than that more remote.

Fig. 97. 98. 99. *Let the Pole of parallel Circles in a Sphere, be in the Circumference of the great Circle AB, which two other great Circles BC, AC, cut at right Angles, and let BC, be the parallel great Circle, and AC, oblique to the Parallels. Assume the non-continuus Arc's DE, FG, equal; and thro' D,E,F,G, let there be drawn the Parallels DH, EI,FK,GL. I say the Arc HI, is greater than the Arc KL. For the intermediate Arc EF, is either*

either commensurable to DE, or FG, or not. First let it be commensurable. (a) And having found V, the greatest common Measure, cut the three Arc's DE, EF, FG, into parts equal to V, and thro' the Points of Division describe Parallels. Therefore because the continuus Arc's DP, PE, EO, &c. are equal; the Arc HT, (b) will be greater than the Arc TI, and TI, greater than IS, &c. Wherefore since HT, is greater than KQ, and TI, than QL, the whole Arc EI, will be greater, than the whole Arc KL, which was proposed. (a) 4. 10. (b) 3. of this.

Now let EF, be incommensurable to DE, or FG. I say the Arc HI, is still greater than KL. For if it be not greater, it will be either lesser or equal. First let it be lesser; and from KL, cut off KM, equal to HI; and thro' M, draw the parallel MN. Moreover by Lemma 2. Prop. 8. *of this Book, find the Arc FO, greater than FN, but lesser than FG, and commensurable to the intermediate Arc EF: And let EP, be equal to FO, (which is lesser than FG, and so also lesser than DE, equal to FG,) and thro' O,P, describe the parallels OR, PQ. Therefore because the non-continuus Arc's PE, FO, are equal, and the intermediate Arc EF, is commensurable to each of them, the Arc QI, (as has been proved just now) will be greater than the Arc KR. And therefore it will be much greater than KM; and accordingly the Arc HI, will be much greater than KM, But HI is supposed equal to KM. Which is absurd. Therefore HI, is not lesser than KL.*

Again, let, if it can be, the Arc HI, be equal to the Arc KL. And having bisected the Arc's DE, FG, in M,N, draw thro' M,N, the Parallels MO, NP. (c) Therefore the Arc HO, will be greater than the Arc OI; and KP, greater than PL. Wherefore OI, will be lesser than half of HI, and KP, greater than half KL. Whence since HI, KL, are supposed equal, OI, will be lesser than KP. Which is absurd. For Because the Arc's EM, FN, (half of the Arc's DE, FG,) are equal, and not continuus, OI cannot be greater than KP, as has been proved. Therefore the Arc HI, is not equal to the Arc KL: But it has been proved not to be lesser. Wherefore it will be greater. Q. E. D. (c) 5. of this.

II.

If a great Circle in the Sphere touches another Circle of the Sphere, and another great Circle is oblique to parallel Circles in the same Sphere, and touches greater Circles than those, which the first proposed great Circle touches, and if their Contact be in the great Circle first proposed; and there are assumed equal Arc's in the Circumference of the oblique Circle, which are not continuus, but yet are on the same Side of the parallel great Circle; and Lastly, if thro' the Points terminating the equal Arc's, parallel Circles are described: These cut off unequal Arc's from the great Circle first proposed, whereof that, which is nigher to the parallel great Circle, will always be greater than that more remote.

This Theorem *is demonstrated from* Prop. 7. *of this Book, in the same manner as the precedent* Theorem *was demonstrated from* Prop. 5. *so that the two great Circles AB, AC, of the precedent* Theorem, *do touch two Parallels, as in* Prop. 7. *of this Book, is said. The rest of the Construction does not differ from the Construction of the Figure of the preceding* Theorem, &c.

III.

If a great Circle in a Sphere, touches another Circle of the Sphere, and some other great Circle oblique to the Parallel Circles, touches greater Circles than those, which the great Circle first proposed touches, and if their Contact be in the great Circle first proposed; and if there are assumed equal Arc's in the Periphery of the oblique Circle, which are not continuus, but yet are on the same Side of the parallel great

great Circle; and if through the Points terminating the equal Arc's great Circles are described, which likewise touch the same Circle, which the first proposed great Circle touches, and intercept similar Arc's of the Parallels, and have those Semicircles which tend from the Points of Contact to the Points terminating the equal Arc's of the oblique Circle, thro' which they are described, so, that they do not meet with that Semicircle of the first proposed great Circle, in which the Contact of the oblique Circle between the apparent Pole and the parallel great Circle is: They cut off unequal Arc's from the parallel great Circle, whereof that nigher the great Circle first proposed, will always be greater than that more remote.

This Theorem is also demonstrated from Prop. 8. *of this Book, like as* Prop. *9. was demonstrated from* Prop. *6: so that the great Circles of* Prop. *9. proceeding from A, do touch a Circle lesser than that which DC ought to touch, &c.*

THEO. X. PROP. X.

If the Pole of parallel Circles in a Sphere be in the Circumference of a great Circle, which two other great Circles cut at right Angles, one of which is one of the Parallels, and the other oblique to the parallels; and if in this oblique Circle any two Points are taken on the same Side of the parallel great Circle; it will be, as the Arc of the parallel great Circle intercepted between the first proposed great Circle, and the

the nearest great Circle described thro' the aforesaid Pole, and one of the Points, is to the Arc of the oblique Circle intercepted between those same Circles; so is the Arc of the parallel great Circle, intercepted between the two great Circles, described thro' the Pole and each of the aforesaid Points, to some other Arc which is lesser than the Arc of the oblique Circle intercepted between the aforesaid Points.

Fig. 100. 101. 102. LET A, the Pole of parallel great Circles in a Sphere, be in the Circumference of the great Circle AB, which two other great Circles BD, CD, cut at right Angles, and let BD, be the parallel great Circle, and CD, oblique to the Parallels; in which having any where taken the two Points E,F, *(a)* describe thro' E, F, and the Pole A, the great Circles AEG, AFH. I say as the Arc BH, is to the Arc CF; so is the Arc HG, to an Arc lesser than FE. For the Arc's CF, FE, are commensurable, or not. First, let them be commensurable, and having *(b)* found P, their greatest common measure, divide the Arc's CF, EF, into Arc's, equal to that greatest Measure, *(c)* and through the Pole A, and the Points of Division, draw the great Circles IM, KN, LO. Therefore because the continuus Arc's CL, LK, KF, FI, IE, are equal, the Arc BO, *(d)* will be greater than the Arc ON, and ON, greater than NH, &c. Therefore the Proportion of BO, to CL, *(e)* will be greater than the Proportion of ON to LK; and the Proportion of OM, to LK, will be greater than of NH, to KF, &c. Wherefore since there are several Mangnitudes, as BO, ON, NH, and the same Number of several others, as CL, LK, KF, and the Proportion of the first BO, to the first CL, is greater than of the Second ON, to the Second LK; and the Proportion of the Second ON, to the Second LK, is greater than of the Third HN, to the Third KF; the Proportion of BH, to CF, *(f)* will be greater than of NH

(a) 30. 1. of this.
(b) 3. 10.
(c) 20. 1. this.
(d) 6. of this.
(e) 8. 5.
(f) 34. 5.

NH to KF; (g) But the Proportion of NH, to KF, is (g) 8. 5.
yet greater than the Proportion of HM, to FI. Therefore the Proportion of BH to CF, is much greater than the Proportion of HM, to FI: (h) But still the Proportion (h) 34. 5. of HM, to FI, is greater than the Proportion of HG, to EF; becauſe the Arc's HM, MG, are equal in Number to the Arc's FI, IE, (i) and the proportion of (i) 8. 5. the firſt HM, to the firſt FI, is greater than the ſecond MG, to the ſecond IE, as has been ſaid. Therefore the Proportion of BH, to CF, is much greater than of HG, to FE. Let it be as BH, to CF; ſo is HG, to P. Therefore the Proportion of HG to P, will be alſo greater than of HG, to FE; (k) and accordingly the Arc P, will (k) 10. 5. be leſſer than the Arc FE. Wherefore as the Arc BH, is to the Arc CF; ſo is the Arc HG, to the Arc P, leſſer than FE. Q. E. D.

But now let the Arc's CF, FE, be incommenſurable. I ſay ſtill, as the Arc BH, is to the Arc CF; ſo is the Arc HG, to an Arc leſſer than EF. For if it be not ſo, it will be, as BH, is to CF; ſo is HG, either to an Arc greater than EF, or to the Arc EF, itſelf. Firſt, let it be, if poſſible, as BH, is to CF; ſo is HG, to the Arc FI, greater than the Arc FE.

Now find by *Lemma* 2. of *Prop* 8. of this Book, the Arc FK, greater than FE, but leſſer than FI, and commenſurable to CF; (l) then draw through K, and the (l) 20. 1. *of this.* Pole A, the great Circle KL. Therefore becauſe the Arc's CF, FK, are commenſurable: It will be, as BH, is to CF; ſo is HL, to an Arc leſſer than FK: But as BH, is to CF; ſo is HG, to FI. Therefore alſo it will be, as HG, is to FI; ſo is HL, to an Arc, leſſer than the Arc FK: And by permutation, as HG, is to HL; ſo is FI, to an Arc leſſer than the Arc FK. But HG, is leſſer than HL. Therefore the Arc FI, will alſo be leſſer than that Arc leſſer than FK, the VVhole than the Part. VVhich is abſurd. Therefore it is not, as BH, is to CF; ſo is HG, to an Arc greater than the Arc FE.

Let it be again, if poſſible, as BH, is to CF; ſo is HG, to FE. The Arc FE being biſected in I, (m) deſcribe through I, and the Pole A, the great Circle IK. (m) 20. 1. *of this.* Therefore becauſe the continuus Arc's FI, IE, are equal, HK, (n) will be greater than KG; and conſequently (n) 6. *of this.* HK, will be greater than half HC. (o) Wherefore the

Proportion (o) 8. 5.

proportion of HK, to FI, will be greater than the Proportion of half HG, to FI: *(p)* But as half HG, is to FI; so is the whole Arc HG, to the whole Arc FE. Therefore also the Proportion of HK, to FI, will be greater than of BH, to FE. But as HG, is to FE; so is BH, to CF. Therefore the Proportion of HK, to FI, will be greater than of BH, to CF; *(q)* and so the Arc HK, to an Arc greater than FI, will be as BH, to CF. Which is absurd. For it was just now proved that the Arc BH, to the Arc CF, cannot be, as the Arc HK, to an Arc greater than FI. Therefore it is not, as BH, is to CF; so is HG, to FE: Neither, as has already been demonstrated, is it, as BH, is to CF; so is HG, to an Arc greater than FE. Therefore as BH, is to CF; so will HG, be to an Arc, lesser than the Arc FE. Q. E. D.

(p) 15. 5.
(q) 10. 5.

CORALLARY.

From hence, it is manifest, that the Arc BH, has a greater Proportion to the Arc CF, than the Arc HG, has to the Arc FE. *(r)* For since it is as BH, is to CF; so is HG, to an Arc, lesser than the Arc FE: *(s)* And the Arc HG, to an Arc lesser than FE, has a greater Proportion than to FE; BH, will also have a greater Proportion to CF, than HG, to FE.

(r) 10. *of this.*
(s) 8. 5.

THEO. XI. PROP. XI.

If the Pole of parallel Circles, in a Sphere, be in the Circumference of a great Circle, which two other Circles cut at right Angles, whereof one is one of the Parallels, and the other oblique to the Parallels; and of another great Circle passing thro' the Poles of the Parallels cuts the oblique Circle between the parallel great Circle, and that Parallel which the oblique Circle touches:

es: The Diameter of the Sphere, has, to the Diameter of the last mentioned Parallel, a greater Ratio, than that Arc of the parallel great Circle intercepted between the great Circle first proposed, and the great Circle passing thro' the Poles of the Parallels, has to the Arc of the oblique Circle intercepted between the same Circles.

LET A, the Pole of parallel Circles in a Sphere, be in the Circumference of the great Circle AB, which two other great Circles BC, DE, cut at right Angles, whereof BC, is the parallel great Circle and BE, oblique to the Parallels touching the Parallel DF. Also thro' the Pole A, let there be described another great Circle AE cutting DE, in the Point E, between BC the parallel great Circle, and the Parallel DF, which the oblique Circle touches: I say the Diameter of the Sphere to the Diameter of the Parallel DF, has a greater Ratio than the Arc BC, has to the Arc DE. For let the right Line AG be the common Section of the Circles AB, AE; and BG the common Section of the Circles AB BC; then AG, BG, will be Semidiameters of them, (*a*) (because great Circles in a Sphere mutually bisect each other) and so of the Sphere, cutting each other in G, the Center of the Sphere, and of the great Circles. Also let DL, be the common Section of the Circles AB, DE, which also will be a Diameter of the Sphere passing thro' G. Again, let DM, be the common Section of the Circles AB, DF: then DM, will be a Diameter of the Circle DF, (*b*) because the Circle AB, passes thro' the Poles of the Parallel DF. Also let FN, CG, be the common Sections of the Circles DF, BC, with the Circle AE. From the Pole A, with the distance AE, describe the Parallel OE, and let OH, EH, be the common Sections of it, with the Circles AB, AE; and then FN, EH, CG, will be Semidiameters of the Circles DF, OE, BC, (*c*) because the great Circle AE bisects them thro' their Poles; and so the common Sections are Diameters meeting the Diameters DM, OH, BG, in

Fig. 103.

(*a*) 11. 1. *of this.*

(*b*) 15. 1. *of this.*

(*c*) 15. 1. *of this.*

the Centers N,H,G. For OH is also a Diameter of the
(d) 15. 1. of this. Circle OE, (*d*) since it bisects the Circle AB, thro' the Pole A. Moreover let EG, be the common Section of the great Circles AE, ED, which also will be a Diameter passing through G, the Center of a Sphere. *Lastly*, let EI, be the common Section of the Circles
(e) 10. 1. of this. DE, OE. (*e*) And because the right Line AG, drawn through the Poles of the parallel OE, is at right Angles to the Plan of the Parallel, and falls in its Center H; the Angle OHG, (from *Def.* 3. *lib.* 11. *Euclid*) in the Triangle GHI, will be a right one; and so the Angle
(f) 19. 1. HGI, will be acute. (*f*) Therefore the Side GI, will be greater, than the Side HI. Cut off the right Line IK, equal to IH. And draw the right Line EK. Again, because each Circle DE, OE, is at right Angles to the
(g) 19. 11. Circle AB; (*g*) EI, their common Section will also be perpendicular to the same; and accordingly (from *Def.* 3. *lib.* 11. *Euclid.*) the Angles EIH, EIK, will be right ones. Therefore because the two Sides EI, IH, of the Triangle EIH, are equal to the two Side EI, IK, of the Triangle EIK, and contain equal Angles, *viz.* right ones, as we have demonstrated, the Angles IHE, IKE,
(h) 4. 1. (*h*) will also be equal. But because the Proportion of the right Line GI, to the right Line IK, is greater than of the Angle IKE, that is, of the Angle OHE, to the
(i) 10. 11. Angle IGE, as by and by we shall demonstrate: (*i*)
(k) 15. 11. And the Angle OHE, is equal to BGC; (*k*) (for the right Lines OH, BG, the common Sections of the parallel Plans, OE, BC, made by the Plan AB, are parallel; as also the right Lines EH, CG, the common Sections of the same Plans, made by the Plan AE) the Proportion of the right Line GI, to the right Line IK, that is, to the right Line IH, will be greater than of the
(l) 33. 6. Angle BGC, to the Angle DGE: (*l*) But as the Angle BGC, is to the Angle DGE; so is the Arc BC, to the Arc DE. Therefore the Proportion of the right Line GI, to the right Line IH; will be greater than of the
(m) 4. 6. Arc BC, to the Arc DE. (*m*) But as GI, is to IH: so
(n) 15. 5. is GD, to DN, that is, (*n*) so is the whole Diameter DL,
(o) 16. 11. to the whole Diameter DM, (*o*) (for DN OH, the common Sections of the parallel Plans DF, OE, made by the Plan AB, are parallel) therefore also the Proportion of DL, the Diameter of the Sphere, to DM, the

Diameter

Diameter of the parallel DF, will be greater than of the Arc BC, to the Arc DE. Q. E. D.

LEMMA.

That the Proportion of the right Line GI, to the right Line IK, is greater than of the Angle IKE, to the Angle IGE, we will prove in the following Theorem.

In every right-angled Triangle, if from one of the acute Angles any how to the opposite Side, be drawn a right Line; the proportion of this Side to its Segment, which is next to the right Angle, will be greater than the proportion of the acute Angle, which the Line drawn makes with the aforesaid Side, to the other acute Angle of the Triangle.

Let EGI be a Triangle, right angled, at I, and let there be any how drawn from the acute Angle GEI, to to the opposite Side GI, the right Line EK. I say the Proportion of the right Line GI, to IK, is greater than of the acute Angle IKE, to the acute Angle IGE. (p) For draw thro' G, the right Line GA, parallel to EK, meeting IE, produced in A. Then because the Angle I, is a right one, the Angle IEG, will be acute, and so AEG, obtuse. (q) Therefore the Side EG, in the Triangle GEI, is greater than the Side GI: but in the Triangle AEG, lesser than the Side AG. Wherefore the Arc of a Circle described from the Center G, with the Distance GE, will cut the right Line GI, produced beyond I, viz. *to B, but the right Line GA, on this Side A, as in C. Therefore because the Triangle GAE, is greater than the Sector GEC, the Proportion of the Triangle GAE, to the Triangle GEI, (r) will be greater than of the Sector GCE, to the Triangle GEI: (s) But there is yet a greater Proportion of the Sector GCE, to the Triangle GEI, than to the Sector GEB; because the Triangle GEI, is lesser than the Sector GEB. Therefore the Proportion of the Triangle GAE, to the Triangle GEI, will be much greater than of the Sector GCE, to the Sector GEB: (t) And accordingly*

Fig. 104.
(p) 31. 1.
(q) 19. 1.
(r) 8. 5.
(s) 8. 5.
(t) 28. 5.

cordingly, by compounding, the Proportion of the Triangle G*AI, to the Triangle* G*EI, will be greater than*
(u) 1. 6. *of the Sector* GC*B, to the Sector* G*EB: (u) But as the Triangle* G*AI, is to the Triangle* G*EI; so is the right Line AI, to the right Line EI;*
(x) Cor. 1. 33. 6. *(x) and as the Sector* GC*B, is to the Sector* G*EB; so is the Angle BGC, to the Angle E*GE. *Therefore the Proportion of AI to to IE, will be greater than of the Angle B*G*A;*
(y) 29. 1. *that is, (y) than of the Angle IKE, to the Angle I*GE:
(z) 2. 6. *(z) But as AI, to IE: so is* G*I, to IK. Therefore also the Proportion of the right Line* G*I, to the right Line IK, will be greater than of the Angle IKE, to the Angle I*GE. Q. E. D.

SCHOLIUM.

In the other Version the following Theorem *is added.*

The same Things being supposed, the Diameter of a Sphere, to the Diameter of that Parallel, described thro' that Point of the oblique Circle, thro' which the great Circle passing thro' the Pole of the Parallels is drawn, has a lesser Ratio, than the Arc of the parallel great Circle intercepted between the first proposed great Circle, and the great Circle passing thro' the Poles of the Parallels, to the Arc of the oblique Circle intercepted between the same Circles.

Fig. 105. *Let the Circles be described (as in Prop. precd.) I say the Diameter of the Sphere to the Diameter of the Parallel* GE, *has a lesser Ratio, than of the Arc BC, to the Arc DE. Let* G*H, BI, be the common Sections of the Circles* GE, *BC, with the Circle AB, which*
(a) 15. 1. of this. *will be Diameters of them, (a) because AB, drawn through their Poles bisects them at right Angles. Therefore BI, will also be a Diameter of the Sphere. And because the Circle DE, is supposed at right An-*
(b) 13. 1. of this. *gles to AB, DE (b) will pass through the Poles of AB.*

AB. In the same manner BC, will pass through the Poles of the same AB, since it is supposed at right Angles to it. Wherefore the Point M, wherein they mutually intersect, will be the Pole of the Circle AB; and accordingly the Segment DEL, which is at right Angles to the Circle AB, is unequally divided in the Point E, wherein the Circles DE, GE, intersect one another, and the lesser Part will be ED: (c) Because the (c) 28. 3.
Arc's MD, ML, are equal, as having (from the Def. *of a Pole) equal Subtenses. (d) Therefore the right* (d) Schol.
Line ED, will be lesser than the right Line EG; and 21. 2. *of this.*
so since the Circle GE, is lesser than the Circle DE, the Arc EG, will be greater than the Arc DE. (e) For (e) Lemma 6. *of this.*
if a right Line equal to the right Line ED, cuts off from the Circle GE, a greater Arc, than the right Line DE, from the Circle DE, much more will the right Line EG, which is greater than ED, cut off a greater Arc, &c. (f) Wherefore the Proportion of the Arc (f) 8. 5.
BC, to the Arc GE, will be greater than to the Arc DE. But because, (g) as the Arc BC, is to the whole (g) 15. 5.
Circumference of the Circle BC; so is the Arc GE, to whole Circumference of the Circle GE because of the Similitude of the Arc's BC, GE; and so by permutation, as the Arc BC, is to the Arc GE; so is the whole Circumference of the Circle BC, to the Circumference of the Circle GE; the Proportion of the Circumference of the Circle BC, to the Circumference of the Circle GE, will also be lesser, than of the Arc BC, to the Arc DE. But as the Circumference of the Circle BC, is to the Circumference of the Circle GE; so is the Diameter BI (which is also a Diameter of the Sphere) to the Diameter GH, as Pappus *has demonstrated, and also we in* Lib. de Circuli Dimensione Archemidis. *Therefore also the Proportion of the Diameter of the Sphere EI, to GH, will be lesser than the Arc BC, to the Arc DE. Q. E. D.*

COROLLARY.

Hence the same things being supposed, the Ratio of the Arc BC, of the parallel great Circle intercepted between the first proposed great Circle; and the great Circle AC, passing thro' the Poles of the parallels, to the Arc DE, of the oblique Circle intercepted between the

the ſame Circles is greater than of Radius, to the Sign of the Arc AD, of the great Circle paſſing thro' the Poles of the parallels; but leſſer than Radius to the Sign of AD, the Arc of the firſt propoſed great Circle intercepted between the Poles of the parallels, and the oblique Circle. For becauſe it has been proved in this *Theorem*, that the Arc BC, to the Arc DE, has a greater Proportion than the Diameter of the Sphere to the Diameter of the Parallel GE; (*h*) but as the Diameter of the Sphere BI, is to GH, the Diameter of the Circle GE; ſo is the Radius BK, to the Semidiameter GN, that is, to the Sign of the Arc AE.

(*h*) 10. 2. *of this.*

Therefore alſo the Ratio of the Arc BC, to DE; will be greater than of the Radius BK, to GN, the Sign of the Arc AE.

(*i*) 11. *of this.*

(*i*) Again, becauſe it has been demonſtrated, that the Ratio of the Arc BC, to the Arc DE, is leſſer than of the Diameter of the Sphere to the Diameter of the parallel DF. (*k*) But as the Diameter of the Sphere BI, is to DF, the Diameter of the parallel DF; ſo is the Radius BK, to DO, the Sign of the Arc AD. Therefore alſo the Proportion of the Arc BC, to the Arc DE, is leſſer than of Radius to the Sign of the Arc AD.

(*k*) 15. 5.

THEO. XII. PROP. XII.

If two great Circles touch ſome one of parallel Circles in a Sphere, and intercept ſimilar Arc's of the parallels, intercepted between the great Circles; and if another great Circle oblique to the parallels, touches greater parallels than thoſe, which the firſt propoſed great Circles touch, and the ſame oblique Circle, cuts the ſaid great Circles in Points poſited between the parallel great Circles, and that Circle which the afore-

ſaid

said great Circles touch: The Diameter of the Sphere, to the Diameter of that Circle, which the oblique Circle touches, has a greater Ratio, than the Arc of the parallel great Circle, intercepted between the first proposed great Circles, to the Arc of the oblique Circle intercepted between the same Circles.

LET the two great Circles AB, CD, in a Sphere, touch the parallel AC, and intercept similar Arc's of the Parallels, intercepted between them; and let another great Circle EF, touch the parallel EG, greater than AC in E, which let be oblique to the parallels, and cut the two first AB, CD, between the parallel great Circle HF, and the parallel AC, in the Points I,K. I say the Ratio of the Diameter of the Sphere,to the Diameter of the parallel EG, is greater than of the Arc BD, to the Arc IK (*a*) For thro' L, the Pole of the parallels, and the Points E,I,K, describe the great Circles LH, LM, LN, and thro' K, the parallel KO, cutting the Circle AB, in P. (*b*) Therefore because the Ratio of the Diameter of the Sphere, to the Diameter of the Circle EG, is greater than of the Arc HM, to the Arc HI; and the ratio of the Arc HM, to EI, (*c*) is greater than MN, to IK; the Ratio of the Diameter of the Sphere to the Diameter of the Circle EG, will also be greater than of the Arc MN, to the Arc IK. And because the Arc PK, is similar to the Arc BD, (from the Hypothesis) (*d*) and the Arc OK, similar to the Arc MN; and the Arc PK, lesser than the Arc OK; the Arc BD, will also be lesser than the Arc MN; (*e*) and accordingly the Ratio of the Arc BD, to the Arc IK, will be lesser than of the Arc MN, to the same Arc IK. Therefore since it has been proved, that the Ratio of the Diameter of the Sphere, is to the Diameter of the Circle EG, greater than the Arc MN, to the Arc IK; therefore the Ratio of the Diameter of the Sphere to the Diameter of the Circle EG, will be much greater than of the Arc BD, to the Arc IK. Q. E. D.

Fig. 106.

(*a*) 20. 1. *of this.*

(*b*) 11. *of this.*

(*c*) Cor. 1. *of this.*

(*d*) 10. 2. *of this.*

(*e*) 8. 5.

SCHO-

SCHOLIUM.

In the Greek *Copy it is affirmed that the Ratio of the Diameter of the Sphere, to the Diameter of the Circle EG, is greater than of the Arc BD, to the Arc IK: Which is clearly manifest from our Demonstration. For since the Diameter of the Sphere has a greater Ratio to the Diameter of the Circle EG, than of the Arc BD, to the Arc IK; double the Diameter of the Sphere will have a much greater Ratio to the Diameter of the Circle EG, than the Arc BD, has to the Arc IK; (f) since that double the Diameter of the Sp'ere, to the Diameter of the Circle EG has a greater Ratio then the Diameter of the Sphere to the Diameter of the same Circle EG.*

(f) 8. 5.

THEO. XIII. PROP. XIII.

If parallel Circles in a Sphere intercept equal Arc's of some great Circle on each Side the Point, in which the great Circle cuts the parallel great Circle; and if thro' the Points terminating the equal Arc's, and the Poles of the Parallels be described great Circles, or if great Circles be described touching one of the Parallels, they cut off equal Arc's from the parallel great Circle.

Fig. 107. 108. LET the parallel Circles CD, EF, in the Sphere AB, cut off from the great Circle HF, two equal Arc's GC, GF, on each Side the Point G, in which the Circle HF, cuts the parallel great Circle BG; and thro' the Points C, G, F, draw great Circles either through the Poles of the parallels, as in the first Figure, or touching one and the same parallel, as in the last, cutting the parallel great Circle in H,I. I say the Arc's GH, GI, are equal.

equal. For because the Arc's GC, GF, are supposed e-
qual, (*a*) the Parallels CD, EF, will be equal. And (*a*) 17. 2.
(*b*) therefore also the Arc's GK, GL, will be equal. *of this.*
(*c*) Wherefore right Lines, as CK, FL, will be equal; (*b*) 18. 2.
(*d*) and accordingly in equal Circles CD, FE, they cut *of this.*
off equal Arc's CK, FL; and so the Arc's CK, FL, will (*c*) 3. *of this.*
be similar between themselves: (*e*) But the Arc GH, (*d*) 28. 3.
is similar, to the Arc CK, and the Arc GI, to the Arc (*e*) 10. 2.
FL. Therefore also the Arc's GH, GI, are similar be- *of this.*
tween themselves; and since they be in the same Circle
they are equal between themselves. *Q. E. D.*

SCHOLIUM.

Hence also is manifest, the same things being supposed, that all the Arc's of great Circles intercepted between the Parallels, are equal between themselves, as are CH, HE, KG, GL, DI, IF. For since the Arc's GC, GH, *are equal to the Arc's* GF, G*I*, (*f*) *right Lines CH, FI*, (*f*) 3. *of this.*
are equal; (*g*) *and accordingly also the Arc's CH, FI, will be equal*: (*h*) *But the Arc's* KG, *DI, are equal to* (*g*) 28.
the Arc CH, and the Arc's LG, *EH, to the Arc FI.* (*h*) 10. 2. *of this.*
Therefore all these six Arc's will be equal.

THEO. XIV. PROP. XIV.

If a great Circle in a Sphere touches two parallel Circles, and some other great Circle oblique to them touches two Parallels greater than the former ones; they cut off from the Parallels unequal Arc's, whereof those that be nigher to either of the Poles be too big to be similar to those more remote.

LET the great Circle AB in a Sphere, touch the Circle Fig. 109.
AC; and another great Circle DE, touch the Circle
F, and cut the two Parallels GH, BI, in KE. I say
the

the Arc's KH, EI, are unequal, and KH, which is nigher to the conſpicuous Pole, is too big to be ſimilar to the Arc EI, more remote; or EB, nigher to the occult Pole, is to big too be ſimilar to the Arc KG, more remote. (*a*) For thro' the Points E,K, deſcribe the great Circle LE, CN, touching the Circles AC, ſo that the Semicircles proceeding from C, thro' N, and from A, thro' B, may not meet: As likewiſe the Semicircles from L, thro' E, and from A, thro' I. (*b*) Therefore the Arc's MH, EI, will be ſimilar. Wherefore KH, is too big to be ſimilar to EI. In the ſame manner becauſe BN, GK, are ſimilar, BE, nigher to the occult Pole, will be too big to be ſimilar to the Arc GK, more remote. Q. E. D.

(*a*) 15. 2. *of this.*

(*b*) 13. 2. *of this.*

FINIS.

ERRATA.

PAge 4, Line 24, for *A*, read *E*. p. 7 l. 12, r. *as G*. p. 8. l. 16, dele *Common*. p. 10, *l.* 23. for *Semidiameter* r. *Semidiameters*. p. 17, l. 14, *for it* r. *is*. p. 18, l. 12, for *AE*; r. *AC*. p. 20, l. 24, for *BE*, r. *DE*. p. 22, l. 4, for *another*, r. *the other*. p. 24, l. 26, r. *a Square*. p. 28, l. 33, for *ADC*, r. *ACD*. p. 33, r. (*d*) *Schol* 8 *of this*. p. 40, l. 13, r. *But*. p. 41, l. 16, for *E*, r. *F*. p. 47, l. 12, inſtead of *D* r. *E*. ibidem, *l.* 19, r. *If*. p. 48, l. 32, for *E*, r. *F*. p. 49, l. 21, inſtead of *DE*, r. *GI*. p. 52, l. 24, for *I*, r. *T*. p. 55, in the Margin, r. (*a*) 20. 1. *of this*. p. 56, l. 10, dele (. p. 59, l. 19, for *it*, r. *them*. ibidem, l. 32, for *from I thro' G*, r. *thro' H*. p. 60, l. 19, for *either*. r. *both*. p. 61, l. 15, r. *CEFD*. p. 79. l. 38, for *EN*, r. *EV*. p. 80, l. 2, for *MK*, r. *MP*. p. 90, l. 25, for *to* r. *two*. p. 92, l. 35, for *following*, r. *falling*. p. 97, l. 9, for *Sphera*, r. *Sphæra*. p. 110, l. 12, dele *great*. p. 117, for *Archemidis*, r. *Archimedis*.